University of Nevada, Reno

Topology Counterparts in C*-Algebras

A thesis submitted in partial fulfillment of the
requirements for the degree of
Master of Science in Mathematics.

by

Daniel Surrell Corder

Dr. Bruce Blackadar / Thesis Advisor

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Bruce Blackadar, Ph.D., Advisor

Stanislav Jabuka, Ph.D., Committee Member

Katherine McCall, Ph.D., Graduate School Representative

Marsha H. Read, Ph. D., Dean, Graduate School

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Abstract

This paper describes a contravariant category equivalence between the category of unital commutative C*-algebras with unital *-homomorphisms and the category of compact Hausdorff spaces with continuous functions in order to characterize semiprojective C*-algebras. Results preliminary to establishment of the equivalence yield homeomorphisms between any compact Hausdorff space X, the space of maximal ideals on $C(X)$ endowed with the hull-kernel topology, and the space of characters on $C(X)$ under the weak* topology. The functional calculus herein constructed provides a link between normal elements of a C*-algebra and continuous functions on the spectra of the elements. The equivalences established, along with the functional calculus, provide a means to develop the C*-algebra theory of semiprojectivity by analogy to the topological concept of absolute neighborhood retracts on compact metrizable spaces; the analogy yields many examples of semiprojective C*-algebras. Semiprojectivity theory is an instance of extending well-established consequences from one mathematical context for use in another context via category equivalence and it additionally motivates an exploration of the extent to which results from one context can be developed analogously in the other beyond the limits of the equivalence.

For my love.

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Contents

0 Introduction

The ultimate goals of this paper are to characterize the C*-algebra theory of semiprojectivity as it relates to the topological concept of ANRs via a contravariant category equivalence and to delineate known semiprojective C*-algebras. To this end, a preliminary portion of results presented constructs homeomorphisms between any compact Hausdorff space X, the space of maximal ideals on $C(X)$, and the space of characters on $C(X)$.

The first section outlines basic definitions and results concerning C*-algebras. The section explores special properties of the spectrum and of ideals in a C*-algebra, and it describes the important C*-algebra $C(X)$. Additionally, important C*-algebra element types and major fundamental results concerning C*-algebras are herein included.

Results in the second section outline the construction of the hull-kernel topology on $Prim(A)$, the set of primitive ideals of a C*-algebra A. The results culminating the section establish for any compact Hausdorff space X a homeomorphism between the space $Prim(C(X))$ endowed with the hull-kernel topology and X under its original topology.

Theory in the third section develops the weak* topology and associates the character space $\Omega(A)$ of a C*-algebra A with this topology. Another homeomorphism is established between any compact Hausdorff space X endowed with its original topology and the character space $\Omega(C(X))$ under the weak* topology. Next, a contravariant category equivalence between the category of unital commutative C*-algebras with unital *-homomorphisms and the category of compact Hausdorff spaces with continuous functions. The section concludes with the functional calculus and polar decomposition, tools used in characterizing semiprojective C*-algebras.

The final section establishes an analogy between semiprojectivity and ANRs using the equivalences developed prior. Partial liftings of specified element types in a C*-algebra lead to a collection of examples of semiprojective C*-algebras formed as universal C*-algebras each generated by a set of partially liftable elements. In this way, semiprojectivity theory explores the analogy between semiprojective C*-algebras and ANRs.

1 Fundamental C*-Algebra Theory

This section introduces C*-algebras and delineates basic properties of these algebras. The spectrum of an element is defined, and facts concerning the spectrum and ideals in C*-algebras are here developed as a foundation for the main results of later sections.

1.1 C*-Algebras

An *algebra* is a vector space A over a field $\mathbb{K}$, with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, together with a map $M : A^2 \to A$, $M(a_1, a_2) \mapsto a_1 a_2$ such that

1. $a_1(a_2 a_3) = (a_1 a_2) a_3$,
2. $a_1(a_2 + a_3) = a_1 a_2 + a_1 a_3$,
3. $(a_1 + a_2) a_3 = a_1 a_3 + a_2 a_3$,
4. $\alpha(a_1 a_2) = (\alpha a_1) a_2 = a_1(\alpha a_2)$,

for all $a_1, a_2, a_3 \in A$ and $\alpha \in \mathbb{K}$. A subalgebra is a vector subspace B of an algebra A such that $b_1 b_2 \in B$ for all $b_1, b_2 \in B$. Hereafter, the field $\mathbb{K}$ will always be $\mathbb{C}$.

A *normed algebra* is an algebra A having a norm $\|\cdot\|$ on its vector space structure with the property $\|a_1 a_2\| \leq \|a_1\| \|a_2\|$ for all $a_1, a_2 \in A$. If the norm on A is complete in its norm space structure then A is a *Banach algebra*. An algebra A in which $a_1 a_2 = a_2 a_1$ for all $a_1, a_2 \in A$ is a *commutative* algebra. If a normed algebra A has an element 1 such that $\|1\| = 1$ and $1a = a1 = a$ for all $a \in A$ then A is *unital*; the element 1, which is necessarily unique, is called the *identity* (or *unit*) of A. The condition $\|a_1 a_2\| \leq \|a_1\| \|a_2\|$ for all a_1, a_2 in a normed algebra A guarantees that the multiplication operation $M(a_1, a_2) \mapsto a_1 a_2$ is jointly continuous.

An *involution* on an algebra A is a map $* : A \to A$ defined by $* : a \mapsto a^*$ such that

1. $(a_1 a_2)^* = a_2^* a_1^*$,
2. $(a_1 + a_2)^* = a_1^* + a_2^*$,
3. $(\alpha a)^* = \bar{\alpha} a^*$,
4. $(a^*)^* = a$,

for all $a, a_1, a_2 \in A$ and all $\alpha \in \mathbb{C}$. An algebra A over $\mathbb{K} = \mathbb{C}$ together with the involution $*$ is called a **-algebra*. If A is a Banach algebra that has an involution $*$ with the property $\|a^*\| = \|a\|$ for all $a \in A$, then A is a *Banach *-algebra*. A Banach *-algebra such that $\|a^* a\| = \|a\|^2$ for all $a \in A$

is called a *C*-algebra*. The condition $\|a^*a\| = \|a\|^2$ for all $a \in A$ is called the C*-property of A. A closed subalgebra B of a C*-algebra A is a C*-subalgebra if $b^* \in B$ for all $b \in B$. If S is any subset of a C*-algebra A, the C*-algebra *generated* by S is the smallest C*-algebra $B \subseteq A$ such that $S \subseteq B$; B is denoted by $C^*(S)$. In particular, $C^*(a)$ is the C*-algebra generated by a single element $a \in A$. In a unital C*-algebra A, $1^* = 11^* = (11^*)^* = 1^{**} = 1$, and 1 automatically has norm 1 since $\|1\| = \|1^*1\| = \|1\|^2$ in A and $\|1\| \neq 0$.

The following spaces are important examples of algebras needed in content following. $C(X)$, with X a compact Hausdorff space, is the primary C*-algebra considered in the results that follow.

Example. $\mathbb{C}$*, the complex numbers.* $\mathbb{C}$ is a unital commutative C*-algebra with involution * given by complex conjugation $\lambda^* = \overline{\lambda}$ for $\lambda \in \mathbb{C}$. A subsequent result shows that a Banach algebra in which every non-zero element is invertible is isomorphic to $\mathbb{C}$.

Example. $C(X)$*,* $C_b(X)$*, and* $C_0(X)$*.* Let X be a topological space. The set $C_b(X)$ of all bounded continuous complex-valued functions on X is a unital Banach algebra under the pointwise operations

$$(f+g)(x) = f(x) + g(x),$$

$$(fg)(x) = f(x)g(x),$$

$$(\alpha f)(x) = \alpha f(x),$$

and norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

If X is compact, $C_b(X) = C(X)$, the set of continuous complex-valued functions on X. If X is a locally compact Hausdorff space, the set $C_0(X)$ of continuous functions vanishing at infinity, that is the set of functions such that $W_\epsilon = \{x \in X \mid |f(x)| \geq \epsilon\}$ is compact for every $\epsilon > 0$, is a Banach algebra since it is a closed subalgebra of $C_b(X)$. If X is compact, then $C_0(X) = C(X)$, a unital Banach algebra.

Suppose that X is a topological space, and define an involution * on $C_b(X)$ by $f^* = \overline{f}$. Then $C_b(X)$ is a C*-algebra. Similarly, if X is a locally compact Hausdorff space and $C_0(X)$ has involution $f^* = \overline{f}$, then $C_o(X)$ is a C*-algebra. If X is compact Hausdorff then $C(X) = C_b(X) = C_0(X)$ is a unital commutative C*-algebra.

Example. $B(X)$*, the operators on* X*, and* $M_n(\mathbb{C})$*, the* $n \times n$ *matrices.* Let X be a normed vector space. Then $B(X)$, the set of all bounded linear operators on X, is a normed algebra with addition and scalar multiplication defined pointwise, multiplication defined by

$$(ST)(x) = (S \circ T)(x) = S(T(x)),$$

and norm given by

$\|T\| = \underset{x\neq 0}{sup}\frac{\|T(x)\|}{\|x\|} = \underset{\|x\|\leq 1}{sup}\|T(x)\|.$

If X is a Banach space, $B(X)$ is a Banach algebra, and if H is a Hilbert space, $B(H)$ is a C*-algebra with involution the adjoint operation * defined by $< T^*x, y >=< x, Ty >$ for all $x, y \in H$. $B(X)$ is never commutative unless $dim(X) = 1$. $M_n(\mathbb{C})$, the vector space of $n \times n$ matrices with entries in $\mathbb{C}$, is identified with $B(\mathbb{C}^n)$ and is therefore a unital C*-algebra.

1.2 Ideals and Quotient Algebras

Let A be an algebra. A vector subspace I of A is a *left ideal* if $as \in I$ for all $a \in A$ and $s \in I$ and a *right ideal* if $sa \in A$ for all $a \in A$ and $s \in I$; I is an *ideal* if it is both a left ideal and a right ideal. A *maximal* ideal is a proper ideal I in A that is not contained in any other proper ideal. Zorn's lemma establishes that every proper modular ideal is contained in a maximal ideal, so any unital algebra posesses maximal ideals [8]. An ideal I containing an element u of A such that $a - au \in I$ and $a - ua \in I$ for all $a \in A$ is called a *modular* ideal. If A is a unital algebra with unit 1 then any ideal I in A is modular since $a - a1 = a - 1a = a - a = 0 \in I$ for any ideal I.

A *homomorphism* is a linear map $\phi : A \to B$, where A and B are algebras, having the property that $\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all $a_1, a_2 \in A$. A homomorphism $\phi : A \to B$ is *unital* if both A and B are unital and $\phi(1) = 1$. For any homomorphism $\phi : A \to B$, $\phi(A)$ is a subalgebra of B and $ker(\phi)$ is an ideal in A [8]. A *-homomorphism is a homomorphism $\phi : A \to B$ between C*-algebras A and B having the property $\phi(a^*) = (\phi(a))^*$ for all $a \in A$.

Let I be an ideal of an algebra A. Then the vector space A/I is an algebra with multiplication operation $(a + I)(b + I) = ab + I$ [8], called the *quotient algebra* of A by I. Moreover, A/I is unital if and only if I is modular [9].

Theorem 1.1. *If A is a C*-algebra and I is a closed ideal in A, then I is closed under the involution * and the quotient algebra A/I is itself a C*-algebra when associated with the quotient norm $\|\cdot\|$ defined by $\|a + I\| = \underset{a'\in I}{inf}\|a + a'\|$.*

Proof. The algebraic properties of A/I follow directly from the algebraic operations of A, and the quotient norm is a complete norm by the properties of the complete norm on A. Let $a_1 + I, a_2 + I \in A/I$ and let $\epsilon > 0$. Then $(\|a_1 + I\| + \epsilon) > \|a_1 + s_1\|$ and $(\|a_2 + I\| + \epsilon) > \|a_2 + s_2\|$ for some $s_1, s_2 \in I$ since $\|a + I\| = \underset{a'\in I}{inf}\|a + a'\|$. Consequently,

$$(\|a_1 + I\| + \epsilon)(\|a_2 + I\| + \epsilon) > \|a_1 + s_1\|\,\|a_2 + s_2\| \geq \|a_1a_2 + a_1s_2 + a_2s_1 + s_1s_2\|$$

by the submultiplicative property of the norm on A. Then

$$(\|a_1 + I\|)(\|a_2 + I\|) = \underset{\epsilon\to 0}{lim}(\|a_1 + I\| + \epsilon)(\|a_2 + I\| + \epsilon) \geq \|a_1a_2 + I\|$$

since $a_1s_2 + a_2s_1 + s_1s_2 \in I$ for any s_1, s_2 corresponding to a fixed $\epsilon > 0$. Thus the quotient norm is submultiplicative and A/I is a Banach algebra.

The ideal I is automatically closed under the involution $*$, and this involution induces an involution on A/I rendering A/I a C*-algebra. Proofs of these two facts are beyond the scope of this paper; see [3]. □

The following theorem is a generalization to algebras of an important result from ring theory. It will not be proven here. See [5], [9] for details.

Theorem 1.2. *Suppose that A is a unital commutative algebra. Then an ideal I in A is maximal if and only if A/I is a field.*

Let A be an algebra with $L \subseteq A$ a modular maximal left ideal. Then the largest ideal $I \subseteq L$ of A is $I = \{a \in A \mid aA \subseteq L\}$, called the *primitive* ideal of A associated to L. The set of primitive ideals of A is denoted by $Prim(A)$. A *prime* ideal is an ideal I in A such that for any ideals J_1 and J_2 of A with $J_1J_2 \subseteq I$, the consequence $J_1 \subseteq I$ or $J_2 \subseteq I$ holds.

In a later section, the set $Prim(A)$ of primitive ideals of a C*-algebra will be endowed with the hull-kernel topology. To this end, in any C*-algebra A, $hull(S)$ is defined to be the set of primitive ideals containing S for any ideal S of A, and $ker(R)$ denotes the intersection of all ideals in a nonempty set R of primitive ideals of A.

Some additional facts about ideals in certain types of algebras are the following; see [9] for proofs of these results. An ideal I in a commutative C*-algebra A is primitive if and only if I is modular maximal. Also, any primitive ideal I in a C*-algebra A is prime. In particular, the set $Prim(A)$ of a unital commutative C*-algebra A is equal to the set of its maximal ideals.

1.3 Spectrum

Suppose that A is a unital algebra and let $a \in A$. Then a is *invertible* if an element b exists such that $ab = ba = 1$. Such an element b is unique, and it is denoted by a^{-1}, the *inverse* of a. The set of invertible elements of A, $Inv(A) = \{a \in A \mid a^{-1} \in A\}$, is a group under the multiplication operation [9]. The *spectrum* of an element $a \in A$ is the set $\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \notin Inv(A)\}$. The notation $\sigma(a)$ for $\sigma_A(a)$ will be used when it is clear that a is being considered as an element of A. In a unital Banach algebra A, the *spectral radius* of an element $a \in A$ is defined to be $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

If an algebra A is nonunital, a unital algebra $\tilde{A}$, known as the unitization of A can always be formed from A by adjoining a unit to A. See [9] for details concerning construction of the unitization of an algebra. Under this construction, A can be identified naturally as an ideal of $\tilde{A}$. Moreover, if A

is a normed algebra, $\tilde{A}$ can itself be made into a normed algebra; in this case, A is a closed algebra of $\tilde{A}$ and $\tilde{A}$ is a Banach algebra if A is [9]. The spectrum of an element a in a nonunital Banach algebra A is defined to be $\sigma_A(a) = \sigma_{\tilde{A}}(a)$ and the spectral radius of a is defined to be $r(a) = \sup_{\lambda \in \sigma_A(a)} |\lambda|$. In this case, 0 is always an element of $\sigma_A(a)$ for any $a \in A$.

The following results are basic properties of invertible elements and spectra of elements in a unital Banach algebra. Proofs or alternative proofs of many of the results in this subsection can be found in [9]. A general result following from Liouville's theorem shows that the spectrum of any element in a unital Banach algebra is nonempty, a consequence summarized in lemma 1.3 below. The Gelfand-Mazur theorem is an important result following directly from the fact that this fact. The following two results are proven in [9].

Lemma 1.3. *Suppose that A is a unital Banach algebra. Then the spectrum $\sigma(a) \neq \emptyset$ for all $a \in A$.*

Corollary 1.4. ***[Gelfand-Mazur theorem]*** *Let A be any unital Banach algebra in which every non-zero element is invertible. Then $A = \mathbb{C}1$.*

Proposition 1.5 also relies on the fact that the spectrum $\sigma(a)$ of an element $a \in A$ is nonempty in any unital Banach algebra A. Lemma 1.6 gives a useful characterization of invertible elements in a unital Banach algebra.

Proposition 1.5. *In a unital Banach algebra A, $\sigma(p(a)) = p(\sigma(a))$ for all $a \in A$ and $p \in \mathbb{C}[z]$.*

Proof. Let $p \in \mathbb{C}[z]$. If p is constant, then $p = \lambda$ for some $\lambda \in \mathbb{C}$, in which case $\sigma(p(a)) = p(\sigma(a)) = \lambda$. So assume p is not constant. By the fundamental theorem of algebra, complex constants $\lambda_0, \lambda_1, ..., \lambda_n$ exists with $\lambda_0 \neq 0$ and $p-\lambda = \lambda_0(z-\lambda_1)...(z-\lambda_n)$ for any $\lambda \in \mathbb{C}$. Hence $p(a)-\lambda = \lambda_0(a-\lambda_1)...(a-\lambda_n)$ and $p(a) - \lambda \in Inv(A)$ if and only if $a - \lambda_i$ is invertible if and only if $a - \lambda_i \notin \sigma(a)$ for $1 \leq i \leq n$. Thus $\lambda \in \sigma(p(a))$ if and only if $\lambda = p(\tilde{\lambda})$ for some $\tilde{\lambda} \in \sigma(a)$, meaning $\sigma(p(a)) = p(\sigma(a))$. ☐

Lemma 1.6. *Let A be a unital Banach algebra. Then $1 - a \in Inv(A)$ and $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$ for any $a \in A$ having $\|a\| < 1$.*

Proof. The series $\sum_{n=0}^{\infty} \|a^n\|$ is convergent since $\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1-\|a\|}$, so $\sum_{n=0}^{\infty} a^n$ is itself convergent. Then

$$\lim_{n\to\infty} [(1-a)(1+...+a^n)] = \lim_{n\to\infty} (1-a^{n+1}) = 1$$

since $\lim_{n\to\infty} \|a\|^n = 0$. Also,

$$\lim_{n\to\infty} [(1-a)(1+...+a^n)] = (1-a)\sum_{n=0}^{\infty} a^n$$

so $(1-a)\sum_{n=0}^{\infty} a^n = 1$. Similarly, $(\sum_{n=0}^{\infty} a^n)(1-a) = 1$. Thus $1-a \in Inv(A)$ and $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$. ☐

Proposition 1.7. *Let A be a unital Banach algebra. Then the set $Inv(A)$ is open in A and $\sigma(a)$ is a closed subset of the disc with radius $\|a\|$ centered at the origin for all $a \in A$.*

Proof. Let $a \in Inv(A)$ and let $a' \in A$ be an element such that $\|a - a'\| < \frac{1}{\|a^{-1}\|}$. Then

$$\left\|a^{-1}a' - 1\right\| = \left\|a^{-1}a' - a^{-1}a\right\| \leq \left\|a^{-1}\right\| \|a' - a\| < \left\|a^{-1}\right\| \frac{1}{\|a^{-1}\|} = 1,$$

so by the previous lemma $a^{-1}a' - 1 \in Inv(A)$. Therefore $a' \in Inv(A)$ and $Inv(A)$ is open because $1, a^{-1} \in A$. Next, let $a \in A$ be arbitrary. Suppose that $\lambda \in \mathbb{C}$ is an element such that $\|a\| < |\lambda|$. Then

$$\left\|\lambda^{-1}a\right\| = \left|\lambda^{-1}\right| \|a\| < \left|\lambda^{-1}\right| |\lambda| = 1,$$

so $1 - \lambda^{-1}a \in Inv(A)$ by the previous lemma. Thus $a - \lambda 1 = -\lambda(1 - \lambda^{-1}a) \in Inv(A)$ so $\lambda \notin \sigma(a)$. Hence $\lambda \in \sigma(a)$ implies that $|\lambda| \leq \|a\|$. The set $\mathbb{C}\backslash\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \in Inv(A)\}$ is the preimage of the function $f : \mathbb{C} \to A$ defined by $f(a) = a - \lambda 1$, which is continuous with image $Inv(A)$, an open set, so $\mathbb{C}\backslash\sigma(a)$ is open. Hence $\sigma(a)$ is a closed subset of the disc with radius $\|a\|$ centered at the origin. □

The following theorem shows that maximal ideals in a unital Banach algebra are closed.

Theorem 1.8. *Suppose that A is a unital Banach algebra and that I is a proper ideal in A. Then $\overline{I}$ is proper as well.*

Proof. If $a \in I$ is an element such that $\|1 - a\| < 1$ then $1 - (1 - a) = a$ is invertible. But then $aa^{-1} = 1 \in A$, meaning $I = A$ contrary to the fact that I is proper. Therefore $\|1 - a\| \geq 1$ for all $a \in I$ so that $1 \notin \overline{I}$. Hence $\overline{I}$ is proper. □

By the above, the closure of a maximal ideal I is a proper closed ideal necessarily containing I, so $I = \overline{I}$. Thus every maximal ideal in a unital Banach algebra is closed.

The following theorem is attributable to Beurling. It yields an explicit formula for the spectral radius of an element in a unital Banach algebra A. See [9] for a proof of the result.

Theorem 1.9. ***[Beurling theorem]*** *Let A be a unital Banach algebra with $a \in A$. Then*

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Thus for any element a in a unital Banach algebra A , $r(a) \leq \|a\|$ by the above result and by the submultiplicative property of the norm on A.

1.4 Special C*-Algebra Elements

Let A be a C*-algebra. The following are important types of elements in A.

1. An element a in A is *self-adjoint* if $a^* = a$. An element p in A is a *projection* if $p^* = p = p^2$.

2. An element u in A is *normal* if $u^*u = uu^*$. Additionally, if $u^*u = uu^* = 1$ then u is a *unitary*.

3. An element s in A is a *partial isometry* if s^*s is a projection. An *isometry* is a partial isometry x such that $x^*x = 1$, and a *co-isometry* is an element y such that $yy^* = 1$.

4. If d in A is self-adjoint with $\sigma(d) \subseteq \mathbb{R}^+$, then d is called *positive*. Denote the set of positive elements of A by A^+ and let $d \geq 0$ mean $d \in A^+$. $d \leq e$ means $e - d \in A^+$.

Every C*-algebra A contains self-adjoint elements: for any $a \in A$,

$$(a^*a)^* = a^*a^{**} = a^*a \in A \text{ and } (aa^*)^* = a^{**}a^* = aa^* \in A,$$

so a^*a and aa^* are self-adjoint elements of A. Let a be any element in a C*-algebra A, and define $b = \frac{1}{2}(a + a^*)$ and $c = \frac{1}{2i}(a - a^*)$. Then a can be written uniquely as $a = b + ic$; in other words, if b' and c' are two self-adjoint elements of A such that $a = b' + ic'$, then $b' = b$ and $c' = c$. See [9] for details.

The next pair of results show that the norms of self-adjoint elements and normal elements in a C*-algebra are equal to the spectral radii of their respective spectrums. Proofs can also be found in [3].

Lemma 1.10. *Let A be a C*-algebra with $a \in A$ self-adjoint. Then $r(a) = \|a\|$.*

Proof. Since a is self-adjoint, $\|a^2\| = \|a^*a\| = \|a\|^2$ and hence $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$ by induction. Thus

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

□

Theorem 1.11. *Suppose that A is a C*-algebra and let $u \in A$ be normal. Then $r(u) = \|u\|$.*

Proof. The element u is normal so $u^*u = uu^*$ and

$$\begin{aligned}(r(u))^2 &= \inf_{n\geq 1} \|u^n\|^{2/n} = \inf_{n\geq 1} \|(u^n)^*(u)^n\|^{1/n} = \inf_{n\geq 1} \|(u^*u)^n\|^{1/n} \\ &= r(u^*u) = \|u^*u\| = \|u\|^2,\end{aligned}$$

since u^*u is self-adjoint. Thus $r(u) = \|u\|$. □

Here is an alternative characterization of partial isometries in a C*-algebra.

Proposition 1.12. *An element s in a C*-algebra A is a partial isometry if and only if $s = ss^*s$.*

Proof. Suppose first that $s = ss^*s$. Then

$$(s^*s)^2 = s^*(ss^*s)s^*(ss^*s)s^*(ss^*s) = s^*(ss^*s)s^*s = s^*(ss^*s) = s^*s,$$

so s^*s is a projection and s is a partial isometry.

Conversely, suppose that s is a partial isometry. Let $a = s - ss^*s$. Then

$$a^*a = (s^* - s^*ss^*)(s - ss^*s) = s^*s - s^*ss^*s - s^*ss^*s + s^*ss^*ss^*s = 0$$

since $(s^*s)^2 = s^*s$. Thus $\|a\|^2 = \|a^*a\| = 0$ by the C*-property so that $s = ss^*s$. □

Corollary 1.13. *Let s in a C^*-algebra A be a partial isometry. Then ss^* is a projection.*

Proof. By the previous proposition since $s = ss^*s$,

$$(ss^*)^2 = (ss^*s)s^* = ss^*,$$

from which it is clear that ss^* is a projection. □

For a partial isometry s, s^*s is the *source projection* of s and ss^* is the *range projection* of s.

2 Hull-Kernel Topology on $Prim(A)$

Here the hull-kernel topology on the set of primitive ideals of a C*-algebra is described. The results following establish an equivalence between any compact Hausdorff space X and the set $Prim(C(X))$ of maximal ideals of $C(X)$.

2.1 General Topology Preliminaries

The topology results collected here are used in the establishment of equivalences between a topological space X and spaces of objects on the C*-algebra $C(X)$ associated to X. Moreover the results are used in the development of semiprojective C*-algebras. They are here stated for ease of reference. A fundamental topology text can be referenced for proofs of the results found in this section; see for example [8].

The following theorem is fundamental but it is essential in establishing major results to follow.

Lemma 2.1. *Let $f : X \to Y$ be a bijective continuous function between the topological spaces X and Y. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

The next result about normal spaces is easily proved. It connects consequences of the Urysohn Lemma and the Tietze Extension Theorem to compact Hausdorff spaces.

Lemma 2.2. *Every compact Hausdorff space is normal.*

The next two theorems are among the most important results in all of elementary topology. See [8] for proofs and discussion of the theorems.

Theorem 2.3. ***[Urysohn lemma]*** *Let X be a normal space and suppose that F_1 and F_2 are disjoint closed subsets of X. Let $[a, b] \subseteq \mathbb{R}$ be any closed interval. Then a continuous map $f : X \to [a, b]$ exists such that $f(x) = a$ for all $x \in F_1$ and $f(x) = b$ for all $x \in F_2$.*

Theorem 2.4. ***[Tietze Extension theorem]*** *Let X be a normal space with closed subspace F, and let $[a, b] \subseteq \mathbb{R}$ be any closed interval. Then for any continuous map $f : F \to [a, b]$, there is a continuous map $\tilde{f} : X \to [a, b]$ extending f. Moreover, for any continuous map $g{:}F \to \mathbb{R}$, there is a continuous map $\tilde{g} : X \to \mathbb{R}$ extending g.*

The concept of complete regularity is also necessary in subsequent work. A *completely regular* space is a space X in which one-point sets are closed and wherein for each point x_0 and each closed set F not containing x_0 there is a continuous function $f_{x_0} : X \to [0, 1]$ such that $f_{x_0}(x_0) = 1$ and $f_{x_0}(x) = 0$ for all $x \in F$. A normal space is completely regular by the Urysohn lemma, and by the lemma preceding the statement of the Urysohn lemma above, a compact Hausdorff space is thus completely regular.

The concept of nets generalizes the concept of sequences in a topological space. A basic result from topology shows that for any set A in a metrizable space X and element $x \in X$, $x \in \overline{A}$ if and only if there is a sequence of points of A converging to x. See, for example, [8]. This sequential criterion for defining closed sets in a metrizable space will be used in working with topologies considered in sections following. However, a compact Hausdorff space may not be metrizable. Nonetheless, a generalization of this sequential criterion holds for nets and closures of sets. To this end, some concepts concerning nets and convergence must be established.

A *directed set* is a nonempty set I with a relation $\preceq$ having the properties

1. $\iota \preceq \iota$ for every $\iota \in I$,
2. $\iota_1 \preceq \iota_3$ whenever $\iota_1 \preceq \iota_2$ and $\iota_2 \preceq \iota_3$ for all $\iota_1, \iota_2, \iota_3 \in I$,
3. there exists $\iota \in I$ such that $\iota_1 \preceq \iota$ and $\iota_2 \preceq \iota$ for every $\iota_1, \iota_2 \in I$.

Let X be a topological space and suppose that I is a directed set. A *net* of points of X is a function $x : I \to X$, denoted $\{x_\iota\}_{\iota \in I}$ where $x_\iota = x(\iota)$. A net $\{x_\iota\}_{\iota \in I}$ is said to *converge* to a point $x \in X$, denoted by $x_\iota \to x$, if for each open set $O \in x$, an index $\iota_O \in I$ exists such that $x_\iota \in O$ for any $\iota \in I$ with $\iota_O \preceq \iota$.

Given this framework of nets, the following proposition establishes a generalization of the sequential criterion for a topology on a metrizable space. See [7] for a proof of the result.

Proposition 2.5. *Let X be a topological space, and let S be a subset of X with $x \in X$. Then $x \in \overline{S}$ if and only if there is a net $\{x_\iota\}_{\iota \in I}$ of points of S such that $x_\iota \to x$.*

Similarly, another generalization to a basic result from topology yields the following theorem, proven in [7].

Theorem 2.6. *Suppose that X and Y are topological spaces and that $\phi : X \to Y$. Then ϕ is continuous if and only if for any net $\{x_\iota\}_{\iota \in I}$ such that $x_\iota \longrightarrow x$, it is true that $\phi(x_\iota) \longrightarrow \phi(x)$.*

The preceding topology results will be important in establishing subsequent conclusions.

2.2 Hull-kernel Topology Construction

The hull-kernel topology is here constructed on the set of primitive ideals of a C*-algebra A. When X is a compact Hausdorff space, $Prim(C(X))$ is the set of maximal ideals of $C(X)$, since in this case $C(X)$ is a unital commutative C*-algebra, and by comments following theorem 1.2 an ideal of $C(X)$ is primitive if and only if it is maximal whenever X is compact Hausdorff.

The next result characterizes the proper closed ideals of a C*-algebra A, and the theorem following yields the hull-kernel topology on $Prim(A)$. Proofs are available in [9].

Lemma 2.7. *Let I be a proper closed ideal in a C*-algebra A. Then $I = ker(hull(I))$.*

Define $\hat{R} = hull(ker(R))$ for $R \subseteq Prim(A)$ and let $\mathcal{C}_{hk} = \{\hat{R} \mid R \in Prim(A)\}$. The next theorem, proven in [9], follows from the facts (also established in [9]) that for sets $F_1, F_2 \in \mathcal{C}_{hk}$, $\widehat{F_1 \cup F_2} = \hat{F}_1 \cup \hat{F}_2$, and that for an arbitrary collection $\{F_\lambda\}_{\lambda\in\Lambda}$ of sets of $\mathcal{C}_{hk}$, $\widehat{\bigcap_{\lambda\in\Lambda} F_\lambda} = \bigcap_{\lambda\in\Lambda} \hat{F}_\lambda$.

Theorem 2.8. *Let A be a C*-algebra. There is a unique topology $\mathcal{T}_{hk}$ on $Prim(A)$ such that $\overline{R} = \hat{R} = hull(ker(R))$ for each $R \subseteq Prim(A)$.*

The unique topology $\mathcal{T}_{hk}$ is $\mathcal{T}_{hk} = \{Prim(A) \setminus \hat{R} \mid R \subseteq Prim(A)\}$, the set of complements of closures of subsets of $Prim(A)$.

The next theorem will be proven in the case of $A = C(X)$, with X a compact Hausdorff space, within the results of this section but the bijective correspondence specified holds for general C*-algebras. The second theorem following establishes a relationship between closed ideals in a C*-algebra and their corresponding hulls. See [9] for proofs of these results.

Proposition 2.9. *Let A be a C*-algebra. The map ϕ from the set of closed ideals of A onto the set of closed subsets of $Prim(A)$ defined by $\phi(I) = hull(I)$ is a bijection.*

Proposition 2.10. *Let A be a C*-algebra. If I and J are closed ideals of A, then $I \subseteq J$ if and only if $hull(J) \subseteq hull(I)$.*

In particular, the preceding results concerning the hull-kernel topology apply to $C(X)$ when X is a compact Hausdorff space.

2.3 Hull-kernel Topology on $Prim(C(X))$

In this section, X is a compact Hausdorff space. A homeomorphism between X and the set $Prim(C(X))$ of maximal ideals on $C(X)$ with the hull-kernel topology is established in the results that follow.

The next lemma and the theorem immediately following provide the technical details and intuition for the connection between closed sets of a compact Hausdorff space X and closed ideals in $C(X)$.

Lemma 2.11. *Let I be a closed ideal in $C(X)$ and define $Y = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$. Then for any open set U containing Y, there is an $f \in I$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \in X \setminus U$.*

Proof. Let $\{y_n\}_n \subseteq Y$ be a sequence that converges to an element $y \in X$. Then $f_I(y_n) = 0$ for all $f_I \in I$, and since f_I is continuous, $0 = f_I(y_n) \to f_I(y)$ so that $f_I(y) = 0$ and $y \in Y$. Hence Y is closed. By the completely regular property of the compact Hausdorff space X, for each $z \in X \setminus U$ a function $f_z \in I$ exists such that $f_z(z) \neq 0$. Let $V_z = \{x \in X \mid f_z(x) \neq 0\}$. By continuity of f_z, the set V_z is open for each $z \in X \setminus U$. The set $\mathcal{V} = \{V_z \mid z \in X \setminus U\}$ is then an open cover of the compact set $X \setminus U$, so $X \setminus U$ has a finite subcover $\{V_{z_k}\}_{k=1}^n \subseteq \mathcal{V}$.

Define a function $g = f_{z_1}\overline{f_{z_1}} + f_{z_2}\overline{f_{z_2}} + ... + f_{z_n}\overline{f_{z_n}}$ on $X \backslash U$. The function g is in I and $g(x) > 0$ for all $x \in X \setminus U$ since if x is in $X \setminus U$, an element $V_{z_k} \subseteq \{V_{z_k}\}_{k=1}^n$ exists with $x \in V_{z_k}$ and $f_{z_k}(x) \neq 0$. The function $h = \frac{1}{g}$ defined on $X \backslash U$ is a continuous real-valued function, and $X \backslash U$ is compact so h has a minimum a and a maximum b over $X \backslash U$. Hence h extends to a continuous function $\tilde{h} : X \to [a, b]$ by the Tietze extension theorem. The function $\frac{1}{g} : X \to [0, \infty]$ is continuous when considered as an extended real-valued function over X. Let $\hat{h} = min(\tilde{h}, \frac{1}{g})$. Then $\hat{h} \in C(X)$, so the function $f = \hat{h}g \in I$ has the properties $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \in X \backslash U$. ☐

Theorem 2.12. *Let X be a compact Hausdorff space and let I be a closed ideal in $C(X)$. Define $Y = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$. Then $I = \{f \in C(X) \mid f(y) = 0 \text{ for all } y \in Y\}$.*

Proof. Let $J = \{f \in C(X) \mid f(y) = 0 \text{ for all } y \in Y\}$ and suppose $h \in I$. Then clearly by definition of Y, $h(x) = 0$ for all $x \in Y$ so $I \subseteq J$.

Next let $g \in C(X)$ be such that $g(y) = 0$ for all $y \in Y$. Define a set $U_n = \{x \mid |g(x)| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. U_n is open by continuity of g and $Y \subseteq U_n$, so a function $f_n \in I$ exists such that $0 \leq f_n(x) \leq 1$ for all $x \in X$ and $f_n(x) = 1$ for all $x \in X \setminus U_n$. Then for all $n \in \mathbb{N}$, $f_n g \in I$ and

$$|(f_n g - g)(x)| = |(g - g)(x)| = 0$$

whenever $x \in X \backslash U_n$, and

$$|(f_n g - g)(x)| = |(f_n - 1)(x) g(x)| = |(f_n - 1)(x)|\,|g(x)| \leq |g(x)| < \frac{1}{n}$$

for all $x \in U_n$ so that $\|f_n g - g\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and all $x \in X$. Hence a sequence $\{g_n\}_{n \in \mathbb{N}} \subseteq I$ defined by $g_n = f_n g$ for $n \in \mathbb{N}$ exists such that $g_n \to g$, meaning $g \in I$ since I is closed. Hence $J \subseteq I$ Therefore $I = \{f \in C(X) \mid f(y) = 0 \text{ for all } y \in Y\}$. ☐

Quotient algebras of $C(X)$ by a closed ideal are described in the next lemma.

Lemma 2.13. *Let I be a closed ideal in $C(X)$. Then $C(X)/I \cong C(Y)$ for the particular closed subset Y in the definition of I.*

Proof. Since I is closed, by the previous theorem $I = \{f \in C(X) \mid f(y) = 0 \text{ for all } y \in Y\}$ for the closed set $Y \subseteq X$ given by $Y = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$. Define a map $\phi : C(X)/I \to C(Y)$ by $\phi(f + I) = f \mid_Y$. This map is a *-homomorphism because restrictions of continuous functions to closed subsets are continuous. Moreover, if $f_Y \in C(Y)$, then by the Tietze extension theorem, a function $f_X \in C(X)$ exists such that f_X is a continuous extension of f_Y. Then $\phi(f_X + I) = f_Y$, so the map ϕ is onto. Clearly, $ker(\phi) = I$ since for any $f \in I$, $f(y) = 0$ for all $y \in Y$ so that$\phi(f + I) = f \mid_Y = 0$, where 0 is the constant function $0 \in C(Y)$. Therefore by the first isomorphism theorem, $A/I \cong C(Y)$. □

The lemma above leads to a proposition characterizing maximal ideals in $C(X)$.

Proposition 2.14. *Let I be an ideal in $C(X)$. Then I is a maximal ideal if and only if a point $x \in X$ exists such that $I = \{f \in C(X) \mid f(x) = 0\}$.*

Proof. Suppose a point $x \in X$ exists such that $I = \{f \in C(X) \mid f(x) = 0\}$. Define a map $\varphi : C(\{x\}) \to \mathbb{C}$ by $\varphi(f) = f(x)$. For any $k \in \mathbb{C}$, a function $\hat{k} \in C(\{x\})$ defined by $\hat{k}(x) = k$ exists so $\varphi(\hat{k}) = \hat{k}(x) = k$ and φ is onto. Suppose next that $f_1, f_2 \in C(\{x\})$ such that $f_1(x) = f_2(x)$. Then $f_1 = f_2$ and φ is 1-1 since x is the only element of $\{x\}$. φ is a unital *-homomorphism since functions in $C(\{x\})$ are continuous, $\overline{f}(x) = \overline{f(x)}$, and $f(1) = 1$. Thus the map φ is a unital *-isomorphism. By the previous lemma, $C(X)/I \cong C(\{x\}) \cong \mathbb{C}$ and since $\mathbb{C}$ is a field, I is a maximal ideal.

Every maximal ideal is closed so $I = \{f \in C(X) \mid f(y) = 0 \text{ for all } y \in Y\}$ for the closed set $Y = \{y \in X \mid f(y) = 0 \text{ for all } f \in I\}$. Y is nonempty since $I = C(X)$ if $Y = \emptyset$, contrary to maximality of I. By the previous lemma, $C(X)/I \cong C(Y)$. But I is maximal so $C(Y) \cong C(X)/I \cong \mathbb{C}$. If Y contains two or more distinct points then $y_1, y_2 \in Y$ exist such that $y_1 \neq y_2$. But by the Urysohn lemma, a continuous function $f \in C(Y)$ exists such that $f(y_1) = \alpha_1$ and $f(y_2) = \alpha_2$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq \alpha_2$. But then $C(Y)$ can't be *-isomorphic to $\mathbb{C}$ since $\mathbb{C}$ is isomorphic to a space of constant functions and $C(Y)$ contains a nonconstant function. The conclusion $Y = \{x\}$ for some $x \in X$ follows and therefore $I = \{f \in C(X) \mid f(x) = 0\}$. □

Finally, the homeomorphism between X and $Prim(C(X))$ is established below.

Theorem 2.15. *Define a map $\phi : X \to Prim(C(X))$ by $\phi(x) = I_x$ for all $x \in X$, where $I_x = \{f \in C(X) \mid f(x) = 0\}$. Then ϕ is a homeomorphism.*

Proof. By the results preceding, $I \in Prim(C(X))$ iff $I = I_x = \{f \in C(X) \mid f(x) = 0\}$ for some $x \in X$. Thus $x \in X$ is an element such that $\phi(x) = I$, meaning ϕ is onto. Suppose next that x_1 and x_2 are two elements of X such that $\phi(x_1) = \phi(x_2)$. Then $I_{x_1} = I_{x_2}$, which implies that $x_1 = x_2$ since if $x_1 \neq x_2$ by the Urysohn lemma some $g \in C(X)$ exists such that $g(x_1) = 0$ and $g(x_2) \neq 0$ contrary to $I_{x_1} = I_{x_2}$. Thus ϕ is 1-1.

The map ϕ is thus a bijection between X and $Prim(C(X))$, so a topology homeomorphic to the hull-kernel topology on $Prim(C(X))$ exists on X. Denote this topology by $\mathcal{T}_{hk}^{X}$ and the original topology of X by $\mathcal{T}$. ϕ is a homeomorphism between $Prim(C(X))$ with the hull-kernel topology and X with its natural topology if for any set $S \subseteq X$, $x \in \overline{S}^X$ if and only if $x \in \overline{S}^{hk}$, where $\overline{S}^X$ is the closure of S in $\mathcal{T}$ and $\overline{S}^{hk}$ is the closure of S in $\mathcal{T}_{hk}^{X}$. Let $S \subseteq X$, and define $I_S = \bigcap_{I \in \phi(S)}$. Then $I_S = \{f \in C(X) \mid f(x) = 0 \text{ for all } x \in S\}$ since $f(x) = 0$ for all $f \in I_x$ and any $x \in S$, and

$$\overline{\phi(S)} = hull(ker(S)) = \{I \in Prim(C(X)) \mid I \supseteq I_S\},$$

where $\overline{\phi(S)}$ is the closure of $\phi(S)$ in the hull-kernel topology on $Prim(C(X))$. Note that for an element $p \in X$, $p \in \overline{S}^{hk}$ if and only if for any function $f \in C(X)$ such that $f(x) = 0$ for all $x \in S$, $f(p) = 0$. This holds since $p \in \overline{S}^{hk}$ if and only if $I_p \in \overline{\phi(S)}$ if and only if $I_p \supseteq I_S$ so that $I_p = \{f \in C(X) \mid f(p) = 0\}$ contains all $f \in C(X)$ such that $f(x) = 0$ for all $x \in S$.

So let $p \in \overline{S}^X$. Then some net $\{p_\iota\}_{\iota \in \Lambda}$ exists such that $p_\iota \to p$ in the topology $\mathcal{T}$ on X, and by a general property $f(p_\iota) \to f(p)$ for any $f \in C(X)$. Since $f(p_\iota) = 0$ for all $p_\iota \in \{p_\iota\}$, $f(p) = 0$ as well. In particular, if $f \in C(X)$ is an element such that $f(x) = 0$ for all $x \in S$, meaning $f \in I_S$, then $f(p) = 0$. Therefore $I_p \supseteq I_S$ so $I_p \in \overline{\phi(S)}$, which leads to the conclusion $p \in \overline{S}^{hk}$.

Conversely, suppose $p \notin \overline{S}^X$. Since X is compact Hausdorff, it is normal and hence completely regular. Thus a function $f_p : X \to [0,1]$ exists such that $f_p(x) = 0$ for all $x \in \overline{S}^X \supseteq S$ and $f_p(p) = 1$. Therefore $p \notin \overline{S}^{hk}$ since $p \in \overline{S}^{hk}$ if and only if for every function $f \in C(X)$ such that $f(x) = 0$ for all $x \in S$, $f(p) = 0$ as well, and f_p is a function not satisfying this condition. Hence by contrapositive, $p \in \overline{S}^{hk}$ implies that $p \in \overline{S}^X$.

The above consequences establishes that ϕ is a homeomorphism. □

The preceding theorem establishes that the original topology on X can be recovered from the hull-kernel topology on $Prim(C(X))$.

3 $\Omega(C(X))$ and the Functional Calculus

In this section, the equivalence between a compact Hausdorff space X and the set of maximal ideals of $C(X)$ is extended to an equivalence with a third object, the set of nonzero homomorphisms on $C(X)$, $\Omega(C(X))$, endowed with the weak* topology. In particular, a homeomorphism exists between $\Omega(C(X))$ and X so that the three spaces X, $\Omega(C(X))$, and $Prim(C(X))$ are all homeomorphic. These results culminate with a category equivalence between compact Hausdorff spaces and unital commutative C*-algebras. The functional calculus is characterized in the last part of this section; the concept is useful in understanding C*-algebras.

3.1 The Weak* Topology

Some preliminaries concerning the construction and properties of the weak* topology are necessary in establishing theory related to the character space of $C(X)$.

Let A be a vector space over a field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A *seminorm* is a function $\sigma : A \to \mathbb{R}$ satisfying the conditions

1. $\sigma(a) \geq 0$ and $\sigma(0) = 0$,
2. $\sigma(\alpha a) = |\alpha|\,\sigma(a)$,
3. $\sigma(a_1 + a_2) = \sigma(a_1) + \sigma(a_2)$,

for all $a, a_1, a_2 \in A$ and $\alpha \in \mathbb{K}$.

Suppose that A is a normed space. Define for each $a \in A$ a function $\sigma_a : A^* \to \mathbb{R}$ by $\sigma_a(\phi) = |\phi(a)|$ for $\phi \in A^*$. Then σ_a is a seminorm on A^* for each $a \in A$ [7]. The ***weak* topology*** on A^*, the dual space of A, is the topology induced by the collection of seminorms $\mathcal{S} = \{\sigma_a \mid a \in A\}$. In other words, the weak* topoloy is the topology having subbasis the sets $\mathcal{S}_{a,\epsilon} = \{\phi \in A^* \mid \sigma_a(\phi) < \epsilon\}$ for all $a \in A$ and $\epsilon > 0$.

Let $\{f_n\}$ be a sequence of bounded linear functionals on a normed space A. Then $\{f_n\}$ *converges weak** if a bounded linear functional f on A exists such that $f_n(a) \longrightarrow f(a)$ for all $a \in A$. Likewise, if $\{f_\iota\}_{\iota \in I}$ is a net of bounded linear functionals on a normed space A, $\{f_\iota\}_{\iota \in I}$ converges weak* in A^* if a bounded linear functional f on A exists such that $f_\iota(a) \longrightarrow f(a)$ for all $a \in A$. By Proposition 2.5, if S is a subset of A^*, then an element $f \in A^*$ is in $\overline{S}$, the closure of S in the weak* topology, if and only if there is a net $\{f_\iota\}_{\iota \in I}$ in S such that $f_\iota \longrightarrow f$ weak*.

Alaoglu's theorem, below, shows that the closed unit ball of the dual space of a C*-algebra is weak* compact. See [7] for a proof of this result.

Theorem 3.1. ***[Alaoglu's theorem]*** *Let A be a normed space with dual space A^*. Then the closed unit ball, $\overline{B_1}(0) = \{f \in A^* \mid \|f\| \leq 1\} \subseteq A^*$, is weak* compact.*

3.2 Character Space on $C(X)$ Equivalence

Let A be a commutative algebra. A *character* on A is a nonzero homomorphism $\phi : A \to \mathbb{C}$. The set of characters on A is denoted by $\Omega(A)$ and is known as the *character space* of A. In this section and beyond, given any unital commutative C*-algebra A, let the dual space A^*, which contains $\Omega(A)$ as a subset, be endowed with the weak* topology. Herein X is a compact Hausdorff space unless otherwise defined. Parallel discussions of proposition 3.2, lemma 3.3, theorem 3.4, and theorem 3.5 following can be found in [9].

Proposition 3.2. *Suppose that A is a unital C*-algebra and let $\delta \in \Omega(A)$. Then δ is a unital *-homomorphism and $\|\delta\| = 1$.*

Proof. Since A is a C*-algebra, $\delta(1) = [\delta(1)]^2$ and since $\delta(1) \neq 0$, it follows that $\delta(1) = 1$. Hence δ is a unital homomorphism. So let $a \in A$. Then $\delta(Inv(A)) \subseteq Inv(\mathbb{C})$, since $a \in Inv(A)$ implies that a^{-1} exists and $\delta(a)\delta(a^{-1}) = \delta(aa^{-1}) = 1$. Let $\lambda \in \sigma(\delta(a))$. $\delta(a - \lambda 1) = \delta(a) - \lambda 1 \notin Inv(\mathbb{C})$ so $\delta(a - \lambda 1) \notin \delta(Inv(A))$, from which it follows that $a - \lambda 1 \notin Inv(A)$ and $\delta(a) \in \sigma(a)$. Therefore $|\delta(a)| \leq r(a) \leq \|a\|$, which implies that $\|\delta\| = \sup\limits_{\|a\| \leq 1} |\delta(a)| \leq 1$. But since $\|1\| = 1$ in any unital normed algebra and $\delta(1) = 1$ by the preceding, $\|\delta\| = 1$.

δ is moreover a *-homomorphism: if $a \in A$, then $a = b + ic$ with b and c self-adjoint so that, by a result proved later (corollary 3.17), $\delta(b) \in \sigma(b) \subseteq \mathbb{R}$ and $\delta(c) \in \sigma(c) \subseteq \mathbb{R}$, meaning

$$\delta(a^*) = \delta(b - ic) = \delta(b) - i\delta(c) = \overline{\delta(b) + i\delta(c)} = \overline{\delta(b + ic)} = \overline{\delta(a)}.$$ □

The following simple lemma is used in the proof of the theorem following it.

Lemma 3.3. *Suppose that A is a unital commutative C*-algebra and let $\delta \in \Omega(A)$. Then $a - \delta(a)1 \in ker(\delta)$ for all $a \in A$.*

Proof. Let $a \in A$. Then for any $\delta \in \Omega(A)$, $\delta(a) \in \mathbb{C}$ and

$$\delta(a - \delta(a)1) = \delta(a) - \delta(\delta(a)1) = \delta(a) - \delta(a)\delta(1) = \delta(a) - \delta(a) = 0.$$

Hence $a - \delta(a)1 \in ker(\delta)$. □

The next two results hold for the space $C(X)$, where X is a compact Hausdorff space. The notation I_x shall henceforth mean $I_x = \{f \in C(X) \mid f(x) = 0\}$. The next theorem follows immediately as a corollary of proposition 2.14 but the alternative proof below gives insight into the nature of the kernel of a character on $C(X)$.

Theorem 3.4. *Let $x \in X$ and $\delta_x \in \Omega(C(X))$, where $\delta_x \in \Omega(C(X))$ is the character defined by $\delta_x(f) = f(x)$ for all $f \in C(X)$. Then $ker(\delta_x)$ is a maximal ideal of $C(X)$, and*

$$ker(\delta_x) = I_x = \{f \in C(X) \mid f(x) = 0\}.$$

Proof. Suppose $\{x_n\}_n \subseteq ker(\delta_x)$ converges to an element $x \in A$. Then $\delta_x(x_n) = 0$ for all $n \in \mathbb{N}$ so $0 = \delta_x(x_n) \to \delta_x(x)$ implies that $\delta_x(x) = 0$ by continuity of δ_x. Hence $ker(\delta_x)$ is closed. $ker(\delta_x)$ is a proper ideal of $C(X)$ since δ_x is nonzero so that $\delta_x(1) \neq 0$. Moreover,

$$C(X) = ker(\delta_x) + \mathbb{C} = \{f + \lambda \mid f \in ker(\delta_x), \lambda \in \mathbb{C}\}$$

since for every $f \in C(X)$, $f - \delta_x(f)1 \in ker(\delta_x)$ by the previous lemma and $\delta_x(f) \in \mathbb{C}$ so

$$f = (f - \delta_x(f)) + \delta_x(f) \in ker(\delta_x) + \mathbb{C}.$$

Then

$$C(X)/ker(\delta_x) = (ker(\delta_x) + \mathbb{C})/ker(\delta_x) \cong \mathbb{C}.$$

Hence $ker(\delta_x)$ is a maximal ideal of $C(X)$ because $C(X)/ker(\delta_x)$ is a field. Since $ker(\delta_x)$ is a maximal ideal, by proposition 2.14 some $x \in X$ exists such that

$$ker(\delta_x) = \{f \in C(X) \mid \delta_x(f) = 0\} = \{f \in C(X) \mid f(x) = 0\} = I_x.$$ □

For any $x \in X$ the character $\delta_x \in \Omega(C(X))$ defined by $\delta_x(f) = f(x)$ for all $f \in C(X)$ is such that $ker(\delta_x) = \{f \in C(X) \mid f(x) = 0\} = I_x$ is a maximal ideal, so θ in the following theorem makes sense.

Theorem 3.5. *The map $\theta : \Omega(C(X)) \to Prim(C(X))$ defined by $\theta(\delta) = ker(\delta)$ is a bijection. Also, $x \in X$ exists for every $\delta \in \Omega(C(X))$ such that $\delta = \delta_x$, where δ_x is defined by $\delta_x(f) = f(x)$ for all $f \in C(X)$. θ has inverse $\theta^{-1} : Prim(C(X)) \to \Omega(C(X))$ given by $\theta^{-1}(I_x) = \delta_x$.*

Proof. Suppose that δ_1 and δ_2 are elements of $\Omega(C(X))$ such that $ker(\delta_1) = ker(\delta_2)$. Then for any $f \in C(X)$, $f - \delta_2(f)1 \in ker(\delta_2) = ker(\delta_1)$ so $\delta_1(f - \delta_2(f)1) = 0$, which implies that

$$\delta_1(f) = \delta_1(\delta_2(f)1) = \delta_2(f)\delta_1(1) = \delta_2(f)$$

since $\delta_2(f) \in \mathbb{C}$. Thus $\delta_1 = \delta_2$ and θ is 1-1.

Suppose that $I \in Prim(C(X))$. Then some $x \in X$ exists such that $I = I_x = ker(\delta_x) = \theta(\delta_x)$. Thus θ is onto, and hence a bijection.

Since θ is a bijection, some $\delta \in \Omega(C(X))$ exists for any $I_x \in Prim(C(X))$ such that $I_x = ker(\delta)$, and in particular, $I_x = ker(\delta_x)$, so $\delta = \delta_x$. Hence every $\delta \in \Omega(C(X))$ is of the form δ_x for some $x \in X$. Define $\psi : Prim(C(X)) \to \Omega(C(X))$ by $\psi(I_x) = \delta_x$. Then

$$\theta \circ \psi(I_x) = \theta(\delta_x) = ker(\delta_x) = I_x \text{ and}$$

$$\psi \circ \theta(\delta_x) = \psi(ker(\delta_x)) = \psi(I_x) = \delta_x.$$

Therefore $\theta^{-1} = \psi$ is defined by $\theta^{-1}(I_x) = \delta_x$. □

In particular, the above theorem establishes that every maximal ideal in $C(X)$ is the kernel of some character in $\Omega(C(X))$ and that every character in $\Omega(C(X))$ is of the form δ_x for some $x \in X$. Henceforth, the notation δ_x for an element of $C(X)$ will denote the character defined by $\delta_x(f) = f(x)$ for all $f \in C(X)$.

The next results establish that a unital commutative C*-algebra endowed with the weak* topology is compact Hausdorff. Actually, the weak* topology is always Hausdorff, but this more general result is not needed and will not be proven here. When a C*-algebra A is $C(X)$ for some compact Hausdorff space X, then in fact the character space of $A = C(X)$ is homeomorphic with X itself, as shown at the end of this subsection.

Theorem 3.6. *Let A be a unital commutative C*-algebra. Then $\Omega(A)$ with the weak* topology is a compact Hausdorff space.*

Proof. Suppose that $\delta \in \overline{\Omega(A)}$. Then some net $\{\delta_\iota\}_{\iota\in\Lambda} \subseteq \Omega(A)$ exists such that $\delta_\iota \to \delta$. In particular, $\delta_\iota(a) \to \delta(a)$ for all $a \in A$ since $\delta_\iota \to \delta$ weak*. Let $a, a_1, a_2 \in A$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}
&\delta_\iota(a_1a_2) = \delta_\iota(a_1)\delta_\iota(a_2) \to \delta(a_1)\delta(a_2) \text{ and } \delta_\iota(a_1a_2) \to \delta(a_1a_2),\\
&\delta_\iota(a_1 + a_2) = \delta_\iota(a_1) + \delta_\iota(a_2) \to \delta(a_1) + \delta(a_2) \text{ and } \delta_\iota(a_1 + a_2) \to \delta(a_1 + a_2),\\
&\delta_\iota(\alpha a) = \alpha\delta_\iota(a) \to \alpha\delta(a) \text{ and } \delta_\iota(\alpha a) \to \delta(\alpha a),\\
&\delta_\iota(1) = 1 \text{ for all } \iota \in \Lambda \text{ so } \delta_\iota(1) \to \delta(1) \text{ means that } \delta(1) = 1.
\end{aligned}$$

In particular, $\delta \in \Omega(A)$ so $\Omega(A) = \overline{\Omega(A)}$ and $\Omega(A)$ is weak* closed. Since $\|\tau\| = 1$ for any $\tau \in \Omega(A)$, $\Omega(A)$ is thus a weak* closed subset of the closed unit ball $\overline{B_1}(0)$ of A^* under the weak* topology. $\overline{B_1}(0)$ is weak* compact by Alaoglu's theorem, so $\Omega(A)$ is weak* compact as a weak* closed subset of a weak* compact set.

Next, suppose that δ_1,$\delta_2 \in \Omega(A)$ with $\delta_1 \neq \delta_2$. Then $a \in A$ exists such that $\delta_1(a) \neq \delta_2(a)$. $\mathbb{C}$ is Hausdorff, so disjoint neighborhoods $U_1 \ni \delta_1(a)$ and $U_2 \ni \delta_2(a)$ exist. Define

$$\tilde{U_1} = \{\delta \in \Omega(A) \mid \delta(a) \in U_1\} \text{ and } \tilde{U_2} = \{\delta \in \Omega(A) \mid \delta(a) \in U_2\}.$$

Let $(\delta_\iota)_{\iota\in I}$ be a net in $\Omega(A) \setminus \tilde{U_1}$ that converges to an element $\delta \in \Omega(A)$. Then $\delta_\iota(a) \to \delta(a)$ by definition of weak* convergence. Then $\delta_\iota(a) \in \mathbb{C} \setminus U_1$ for each $\iota \in I$ so $\delta(a) \in \mathbb{C} \setminus U_1$ since $\mathbb{C} \setminus U_1$ is a closed set. Hence $\delta(a) \notin U_1$ so $\delta \in \Omega(A) \setminus \tilde{U_1}$, which implies that $\Omega(A) \setminus \tilde{U_1}$ is closed. Thus $\tilde{U_1}$ is open. By the same reasoning, $\tilde{U_2}$ is an open set. Then $\tilde{U_1} \ni \delta_1$ and $\tilde{U_2} \ni \delta_2$ are disjoint neighborhoods and $\Omega(A)$ is Hausdorff. □

Theorem 3.7. *Suppose that X is a compact Hausdorff space. Then there is a homeomorphism between X and the character space $\Omega(C(X))$ given by the map $\varphi : X \to \Omega(C(X))$, $\varphi(x) = \delta_x$, where $\delta_x \in \Omega(C(X))$ is the character defined by $\delta_x(f) = f(x)$ for $f \in C(X)$.*

Proof. Let $\phi : X \to Prim(C(X))$ be the homeomorphism defined by $\phi(x) = I_x$ and let $\theta^{-1} : Prim(C(X))$ be the map given by $\theta^{-1}(I_x) = \delta_x$. Then $\theta^{-1} \circ \phi$ is a bijection since it is a composition of two bijective functions. Moreover,

$$\varphi(x) = \delta_x = \theta^{-1}(I_x) = \theta^{-1} \circ \phi(x)$$

for all $x \in X$ so $\varphi = \theta^{-1} \circ \phi$. Thus φ is a bijection.

Next, let $\delta_x \in \Omega(C(X))$ and let $\{\delta_{x_\lambda}\}_{\lambda\in\Lambda} \subseteq \Omega(C(X))$ be a net converging to δ_x. Then $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq X$ is a net converging to the point $x \in X$ such that x_λ corresponds to δ_{x_λ} for each $\lambda \in \Lambda$. Moreover, by continuity of the elements of $C(X)$, $f(x_\lambda) \longrightarrow f(x)$ for any $f \in C(X)$. Hence $\delta_{x_\lambda}(f) = f(x_\lambda) \longrightarrow f(x) = \delta_x(f)$ for all $f \in C(X)$ so that $\{\delta_{x_\lambda}\}_{\lambda\in\Lambda} \subseteq \Omega(C(X))$ is a net converging weak* to δ_x. Consequently, φ is continuous.

φ is a continuous bijection between compact Hausdorff spaces so φ is a homeomorphism. □

3.3 Unital Commutative C*-Algebra Category Equivalence

The preceding results can now be employed to construct a category equivalence between unital commutative C*-algebras and compact Hausdorff spaces. First some preliminaries characterizing commutative C*-algebras are necessary.

If A is a commutative C*-algebra and $a \in A$, the map $\hat{a} : \Omega(A) \to \mathbb{C}$ defined by $\hat{a}(\delta) = \delta(a)$ is known as the *Gelfand transform* of a. The following Gelfand theorem is one of the most important theorems in the theory of C*-algebras. It shows that a commutative C*-algebra A can be thought of as the C*-algebra $C_0(\Omega(A))$, with the association of $a \in A$ to its Gelfand transform $\hat{a} \in C_0(\Omega(A))$. The theorem's proof will not be given here but it could be pieced together mostly from results presented within this paper; see in particular the results establishing the functional calculus presented in the next subsection. Complete proofs can be found in [3] and [9]. The isomorphism in the Gelfand theorem yields a representation known as the Gelfand representation.

Theorem 3.8. ***[Gelfand theorem]*** *Let A be a non-zero commutative C*-algebra. Then the map $\Gamma : A \to C_0(\Omega(A))$ defined by $\Gamma(a) = \hat{a}$ is an isometric *-isomorphism.*

The following lemma relates continuous functions between compact Hausdorff spaces to unital *-homomorphisms between C*-algebras of continuous functions on the spaces.

Lemma 3.9. *Let X and Y be compact Hausdorff spaces. For every continuous function $f : X \to Y$, there is a unital *-homomorphism $\phi_f : C(Y) \to C(X)$ given by $\phi_f(g) = g \circ f$.*

*Conversely, for every unital *-homomorphism $\phi : C(Y) \to C(X)$, there is a continuous function $f_\phi : X \to Y$ given by $f_\phi(x) = y$, where $y \in Y$ is the unique element that corresponds to $\delta_y = \delta_x \circ \phi \in C(Y)$ under the homeomorphism $\varphi : Y \to \Omega(C(Y))$ given by $\varphi(y_0) = \delta_{y_0}$.*

Proof. Let $f : X \to Y$ be a continuous function and define $\phi_f : C(Y) \to C(X)$ by $\phi_f(g) = g \circ f$. By fundamental properties of compositions of continuous functions, this map is closed under addition, multiplication, scalar multiplication, and complex conjugation. Moreover, $\phi_f(1) = 1 \circ f = 1$. Hence ϕ_f is a unital *-homomorphism.

Conversely, suppose that $\phi : C(Y) \to C(X)$ is a unital *-homomorphism. Let $x \in X$. Then δ_x corresponds to x under the homeomorphism $\varphi_1 : X \to \Omega(C(X))$ given by $\varphi(x_0) = \delta_{x_0}$. Also, $\delta_x \circ \phi = \delta_y$ for some $y \in Y$ and δ_y corresponds to this y under the equivalence $\varphi_2 : Y \to \Omega(C(Y))$ given by $\varphi(y_0) = \delta_{y_0}$. Define in this fashion for each $x \in X$ a function $f_\phi : X \to Y$ by $f_\phi(x) = y$. Suppose that $\{x_\iota\}_{\iota \in \Lambda} \in X$ is a net with $x_\iota \to x$. Then x_ι corresponds to $\delta_{x_\iota} \in C(X)$ and $\delta_{x_\iota} \to \delta_x$ because $x_\iota \to x$. Let $f \in C(Y)$. Then $\phi(f) = g$ for some $g \in C(X)$. Since $\delta_{x_\iota} \to \delta_x$, $\delta_{x_\iota}(g) \to \delta_x(g)$, which means that $\delta_{y_\iota}(f) \to \delta_y(f)$, where $\delta_{y_\iota} = \delta_{x_\iota} \circ \phi$. Then since $f \in C(Y)$ is arbitrary, $\delta_{y_\iota} \to \delta_y$ in the topology on $\Omega(C(Y))$; the topology of Y is homeomorphic to this topology so $f_\phi(x_\iota) = y_\iota \to y = f_\phi(x)$. Consequently, f_ϕ is continuous. □

The above lemma gives a correspondence between continuous functions and unital *-homomorphisms. This correspondence is the basis for the category equivalence given in the following theorem.

Theorem 3.10. *There is a contravariant category equivalence between the category of unital commutative C*-algebras and unital *-homomorphisms with the category of compact Hausdorff spaces and continuous functions.*

Proof. Let A and B be unital commutative C*-algebras. By the Gelfand theorem, $A = C(X)$ and $B = C(Y)$ for the compact Hausdorff spaces $X = \Omega(A)$ and $Y = \Omega(B)$. The lemma preceding this theorem shows that there is a continuous function $f : X \to Y$ if and only if there is a unital *-homomorphism $\phi_f : C(Y) \to C(X)$ given by $\phi_f(h) = h \circ f$.

Suppose that C is any other unital commutative C*-algebra, so that $C = C(Z)$ for the compact Hausdorff space $Z = \Omega(C)$. Let $g : Y \to Z$ be a continuous function and let $\phi_g : C(Z) \to C(Y)$ be its corresponding unital *-homomorphism. The continuous function $g \circ f : X \to Z$ corresponds to a unital *-homomorphism $\phi : C(Z) \to C(X)$ given by $\phi(h) = h \circ (g \circ f)$ for $h \in C(Z)$ by the previous lemma. Let $h \in C(Z)$. Then

$$\phi(h) = h \circ (g \circ f) = (h \circ g) \circ f = \phi_f(h \circ g) = \phi_f(\phi_g(h)) = \phi_f \circ \phi_g(h)$$

which shows that $\phi_f \circ \phi_g$ is the *-homomorphism corresponding to the continuous function $g \circ f$. Hence the contravariant category equivalence has been established. □

3.4 Functional Calculus and Polar Decomposition

The functional calculus and polar decomposition are developed here by making use of an isometric *-isomorphism to essentially apply continuous functions to particular elements of C*-algebras.

The Stone-Weierstrass theorem is essential in extending the result $\sigma(p(a)) = p(\sigma(a))$ for a polynomial p and a normal element a in a C*-algebra A to an analogous result for any continuous function $f \in C(\sigma(a))$. The theorem holds for particular algebras over a compact Hausdorff space X that consist of continuous functions separating points of X which vanish at no point of X. A collection of functions $\mathcal{F} \in C(X)$ is said to *separate points* of X if for any distinct elements $x_1, x_2 \in X$ there is an $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$, and the collection $\mathcal{F}$ is said to *vanish at no point* of X if to each $x \in X$ there corresponds a function $g \in \mathcal{F}$ such that $g(x) \neq 0$. Here is the statement of the theorem for complex continuous functions; see for example [10] for a proof of the result.

Theorem 3.11. ***[Stone-Weierstrass theorem]*** *Let X be a compact Hausdorff space. Suppose $\mathcal{F} \subseteq C(X)$ is an algebra that separates points of X such that $\mathcal{F}$ vanishes at no points of X and $\overline{f} \in \mathcal{F}$ for every $f \in \mathcal{F}$. Then $\overline{\mathcal{F}} = C(X)$.*

In particular, the Stone-Weierstrass theorem establishes that the closure of the set of polynomials with complex coefficients is $C(X)$. This important result is made precise in the following corollary.

Corollary 3.12. *Let X be a compact Hausdorff space and denote the algebra of all polynomials with complex coefficients over X by $\mathcal{P}$. Then $\overline{\mathcal{P}} = C(X)$.*

Proof. The algebra $\mathcal{P}$ separates points of X and $\mathcal{P}$ vanishes at no point of X since $\mathcal{P}$ contains the constant functions; $\mathcal{P}$ is also closed under complex conjugation. Thus by the Stone-Weierstrass theorem, $\overline{\mathcal{P}} = C(X)$. □

The next theorem characterizes the spectrum of an element in a commutative unital Banach algebra as the set of evaluations at the element by characters in the algebra's character space; the characterization is useful in subsequent results. See [9] for a proof of the result.

Theorem 3.13. *Suppose that A is a commutative unital Banach algebra and let $a \in A$. Then*

$$\sigma(a) = \{\delta(a) \mid \delta \in \Omega(A)\}.$$

In the following, for an element a in a C*-algebra A, let $P(\sigma(a)) \subseteq C(\sigma(a))$ denote the *-subalgebra of all polynomials on $\sigma(a)$.

Proposition 3.14. *Let A be a unital C*-algebra and suppose that $a \in A$ is normal. Then there is an isometric unital *-isomorphism $\tilde{\varphi} : C(\sigma(a)) \to C^*(a, 1)$ such that $\tilde{\varphi}(\iota) = a$, where $\iota : \sigma(a) \to \mathbb{C}$ is the inclusion map.*

Proof. Denote the *-subalgebra of A generated by a and 1 by B. Define a map $\varphi : P(\sigma(a)) \to B$ by $\varphi(p) = p(a)$. Then $\varphi(\iota) = \iota(a) = a$. Moreover, $\sigma(a)$ is compact Hausdorff so $P(\sigma(a))$ is an algebra of functions closed under complex conjugation. Also, $\varphi(1) = 1(a) = 1$. Consequently, φ is a unital *-homomorphism. Let $b \in B$. B is generated by a and 1 so $b = \alpha_0 + \alpha_1 a + ... + \alpha_n a^n$, where $\alpha_0, \alpha_1, ..., \alpha_n \in \mathbb{C}$, from which it is clear that $b = p(a)$ for some $p \in P(\sigma(a))$. Thus φ is onto. φ is also clearly 1-1 since if $p_1, p_2 \in P(\sigma(a))$ then $p_1(a) = p_2(a)$ implies that $p_1 = p_2$. Therefore φ is a unital *-isomorphism.

Next, let $p \in P(\sigma(a))$. Then $\varphi(p) = p(a)$ is normal since a is normal and φ is a unital *-isomorphism. Since $p(a)$ is normal,

$$\|p(a)\| = r(p(a)) = sup\{|\lambda| \in \mathbb{C} \mid \lambda \in \sigma(p(a))\} = sup\{|p(\lambda)| \in \mathbb{C} \mid \lambda \in \sigma(a)\} = \|p\|,$$

since $\sigma(p(a)) = p(\sigma(a))$. Therefore φ is also isometric.

The completion of B is $C^*(a, 1)$ and the completion of $P(\sigma(a))$ is $C(\sigma(a))$ by the Stone-Weierstrass theorem. Thus φ can be extended to an isometric unital *-isomorphism $\tilde{\varphi} : C(\sigma(a)) \to C^*(a, 1)$ such that $\tilde{\varphi}(\iota) = a$. □

The map $\tilde{\varphi}$ in the preceding proposition is known as the *functional calculus* at a. Use the notation $f(a)$ to denote the element $\tilde{\varphi}(f)$ for $f \in C(\sigma(a))$. The following proposition defines a functional calculus for self-adjoint elements in a non-unital C*-algebra. The two definitions of the functional calculus are consistent where they overlap. The proposition is not proven here but its proof is similar to that of the preceding proposition.

Proposition 3.15. *Let A be a nonunital C*-algebra and suppose that $a \in A$ is normal. Then there is an isometric unital *-isomorphism $\tilde{\varphi} : C_0(\sigma(a)\backslash\{0\}) \to C^*(a)$ such that $\tilde{\varphi}(\iota) = a$, where $\iota : \sigma(a) \to \mathbb{C}$ is the inclusion map.*

Thus the functional calculus can be defined in a C*-algebra A according to the preceding two propositions depending on whether A is unital or nonunital. In any event, the definition of the functional calculus $\tilde{\varphi}$ at an element $a \in A$ is unambiguous and is denoted by $f(a)$ for $f \in C(\sigma(a))$. The following two results establish important properties of the functional calculus. Note that $C(\sigma(a))$ is generated by 1 and ι, the inclusion map.

Lemma 3.16. *Suppose that A is a unital C*-algebra with $a \in A$ normal, and let $\delta \in \Omega(C^*(a, 1))$. Then for any $f \in C(\sigma(a))$, $\delta(f(a)) = f(\delta(a))$. More generally, if $\phi : A \to B$ is a unital *-homomorphism between A and the unital C*-algebra B, then for any $f \in C(\sigma(a))$, $\phi(f(a)) = f(\phi(a))$. Analogous results hold for nonunital C*-algebras and self-adjoint elements.*

Proof. The C*-subalgebra $\Omega(C^*(a,1))$ is a unital commutative C*-subalgebra since a is normal. Then since $\delta(a) \in \sigma(a)$ and $\iota(a) = \tilde{\varphi}(\iota) = a$,

$$\delta(\iota(a)) = \delta(\tilde{\varphi}(\iota)) = \delta(a) = \iota(\delta(a)) \text{ and}$$

$$\delta(1(a)) = \delta(\tilde{\varphi}(1)) = \delta(1) = 1 = 1(\delta(a)).$$

Thus $\delta(f(a)) = f(\delta(a))$ since $C(\sigma(a))$ is generated by 1 and ι.

The proof of the general result follows in similar fashion. □

Theorem 3.17. ***[Spectral Mapping theorem]*** *The equality $\sigma(f(a)) = f(\sigma(a))$ holds for any normal element a in a unital C*-algebra A and $f \in C(\sigma(a))$. An analogous result holds for nonunital C*-algebras and self-adjoint elements.*

Proof. Let $a \in A$ and let $f \in C(\sigma(a))$. Then $\delta(f(a)) = f(\delta(a))$ for any $\delta \in \Omega(C^*(a,1))$ by the previous lemma, so

$$\sigma(f(a)) = \{\delta(f(a)) \mid \delta \in \Omega(C^*(a,1))\} = \{f(\delta(a)) \mid \delta \in \Omega(C^*(a,1))\} = f(\sigma(a)).$$ □

Example. Let A be a unital C*-algebra and let $a \in A$ be a normal element with $\sigma(a) = S_1 \cup S_2$, where $S_1 \subseteq (-\infty, k)$ and $S_2 \subseteq (k, \infty)$ for $0 < k < 1$. Define a function f by

$$f(t) = \begin{cases} 0 & for\, t \in S_1 \\ 1 & for\, t \in S_2 \end{cases}.$$

Then $f \in C(\sigma(a))$ is a projection since $f^2 = f = \overline{f}$. Consequently, $\tilde{\varphi}(f) = f(a)$ is a projection in A by the functional calculus.

Example. Suppose that A is a unital C*-algebra and let $a \in A$ be positive. The function $f : \sigma(a) \to \mathbb{R}$ defined by $f(t) = t^{\frac{1}{2}}$ is a continuous function on $\sigma(a)$ since $\sigma(a) \geq 0$. Hence $\tilde{\varphi}(f) = f(a) = a^{\frac{1}{2}}$ is a positive element in A. A kth power a^k can be defined in this way for any $k > 0$. Likewise, a $-k$th power a^{-k} can be defined for a for any $k > 0$ provided $0 \notin \sigma(a)$.

The functional calculus yields the following properties of particular elements in a C*-algebra.

Corollary 3.18. *Let A be a C*-algebra and let $a \in A$ be self-adjoint. Then $\sigma(a) \subseteq \mathbb{R}$.*

Proof. Under the functional calculus $\tilde{\varphi}$ at a, an isometric unital *-isomorphism, $\tilde{\varphi}(\iota) = a$. Since a is self-adjoint, ι must be self-adjoint in $C(\sigma(a))$. Consequently, $\sigma(\iota) \subseteq \mathbb{R}$, which implies that

$$\sigma(a) = \iota(\sigma(a)) = \sigma(\iota(a)) = \sigma(\tilde{\varphi}(\iota)) \subseteq \sigma(\iota) \subseteq \mathbb{R}.$$ □

Corollary 3.19. *Suppose that A is a unital C*-algebra and let $u \in A$ be a unitary element. Then $\sigma(u) \subseteq \mathbb{T}$.*

Proof. Since $\tilde{\varphi}(\iota) = u$ under the functional calculus $\tilde{\varphi}$ at u, ι must be unitary in $C(\sigma(u))$. Therefore

$$\sigma(u) = \iota(\sigma(u)) = \sigma(\iota(u)) = \sigma(\tilde{\varphi}(\iota)) \subseteq \sigma(\iota) \subseteq \mathbb{T}.$$ □

Corollary 3.20. *An element d in a C*-algebra A is positive if and only if $d = a^*a$ for some $a \in A$.*

Proof. The element d is positive if and only if $\iota \in C(\sigma(p))$ is positive under the functional calculus $\tilde{\varphi}$ at d, which occurs if and only if $\iota = \overline{f}f$ for some $f \in C(\sigma(d))$. Then

$$d = \tilde{\varphi}(\iota) = \tilde{\varphi}(\overline{f}f) = \tilde{\varphi}(\overline{f})\tilde{\varphi}(f) = [\tilde{\varphi}(f)]^*\tilde{\varphi}(f) = a^*a \text{ for } a = \tilde{\varphi}(f) = f(d) \in A.$$

Conversely, if $d = a^*a$ for some $a \in A$ then d is clearly self-adjoint. Moreover,

$$d = a^*a = \overline{\tilde{\varphi}(\iota)}\tilde{\varphi}(\iota) = \tilde{\varphi}(\overline{\iota})\tilde{\varphi}(\iota) = \tilde{\varphi}(\overline{\iota}\iota) = \tilde{\varphi}(f),$$

for some function $f \in C(\sigma(a))$, and since $f = \overline{\iota}\iota$ is positive, its image d is positive as well. □

The preceding consequences yield useful properties of special elements in the C*-algebra $C(X)$, where X is a compact Hausdorff space.

Proposition 3.21. *The following results results hold for elements of $C(X)$ with X compact Hausdorff.*

(a) $\sigma(f)$ is the range of f for any $f \in C(X)$,

(b) $f \in C(X)$ is self-adjoint if and only if f is real-valued,

(c) $f \in C(X)$ is a projection if and only if f takes only the values 0 and 1,

(d) $f \in C(X)$ is unitary if and only if $|f(x)| = 1$ for all $x \in X$.

Proof. Let $f \in C(X)$. Then f is invertible if and only $f(x) \neq 0$ for all $x \in X$, so if $\lambda \in \mathbb{C}$, $f - \lambda 1$ is invertible if and only if $f(x) - \lambda \neq 0$ for all $x \in X$. Hence $f - \lambda 1 \in Inv(X)$ if and only if $\lambda \notin f(X)$, meaning $\lambda \in \sigma(f)$ if and only if $\lambda \in f(X)$. Therefore $\sigma(f)$ is the range of f.

By corollary 3.18, if $f \in C(X)$ is self-adjoint then $\sigma(f) \subseteq \mathbb{R}$. Thus $\sigma(f) = f(X) \subseteq \mathbb{R}$. If $f(X) \subseteq \mathbb{R}$, then $f = \overline{f}$ so f is self-adjoint.

If f is a projection then $f^2 = f$, which means that f can only take the values 0 and 1. Conversely, if $f(X) \in \{0, 1\}$ then $[f(x)]^2 = 0$ whenever $f(x) = 0$ and $[f(x)]^2 = 1$ whenever $f(x) = 1$ so f is a projection.

By corollary 3.19, if f in $C(X)$ is unitary then $\sigma(f) \subseteq \mathbb{T}$. Hence by the first result of this proof above, $f(X) = \sigma(f) \subseteq \mathbb{T}$, meaning $|f(x)| = 1$ for all $x \in X$. Conversely, if $|f(x)| = 1$ then $f\overline{f} = \overline{f}f = |f|^2 = 1$, so f is unitary. □

The following proposition shows that any invertible element in a unital C*-algebra can be decomposed uniquely as the product of a unitary and a positive element. This property is called *polar decomposition.*

Proposition 3.22. ***[Polar decomposition]*** *Suppose that A is a unital C*-algebra and let $a \in A$ be invertible. Then a can be written uniquely as the product of a unitary u and a positive element d.*

Proof. The element a^*a is positive since a^*a is self-adjoint and $\sigma(a^*a) \geq 0$. Also, a^*a is nonzero since if $a^*a = 0$, then $\|a\|^2 = \|a^*a\| = 0$, which would imply that $a = 0$ contrary to invertibility of a. Define $u = a(a^*a)^{\frac{-1}{2}}$ and $d = (a^*a)^{\frac{1}{2}}$. Then $(a^*a)^{\frac{1}{2}}$ is positive since f defined by $f(t) = t^{\frac{1}{2}}$ is positive on $\sigma(a^*a)$. And since $(a^*a)^{\frac{-1}{2}}$ is positive by the same reasoning, $((a^*a)^{\frac{-1}{2}})^* = (a^*a)^{\frac{-1}{2}}$ so that

$$u^*u = (a^*a)^{\frac{-1}{2}} a^*a(a^*a)^{\frac{-1}{2}} = 1, \text{ and}$$
$$uu^*a = a(a^*a)^{-\frac{1}{2}}(a^*a)^{-\frac{1}{2}}a^*a = a(a^*a)^{-1}(a^*a) \Rightarrow uu^* = uu^*aa^{-1} = aa^{-1} = 1$$

Thus u is a unitary, d is positive, and $a = ud$.

Uniqueness will not be proven here. □

Polar decomposition is preserved under *-homomorphisms, as the following corollary shows.

Corollary 3.23. *Let a be an invertible element in a unital C*-algebra A such that a has polar decomposition $a = ud$, where u is a unitary and d is positive, and suppose that $\phi : A \to B$ is a *-homomorphism between A and the unital C*-algebra B. Then the image $\phi(a)$ has a unique polar decomposition $\phi(a) = \phi(u)\phi(d)$ in B.*

Proof. The element $\phi(a)$ is invertible because $\phi(a)\phi(a^{-1}) = \phi(aa^{-1}) = 1$ and $\phi(a^{-1})\phi(a) = \phi(a^{-1}a) = 1$. Also, $\phi(u)[\phi(u)]^* = \phi(uu^*) = 1 = \phi(u^*u) = [\phi(u)]^*\phi(u)$ so $\phi(u)$ is a unitary in B. Then $\phi(d)$ is positive in B since $\sigma(\phi(d)) \subseteq \sigma(d) \subseteq \mathbb{R}^+$ and $\phi(d) = \phi(d^*) = [\phi(d)]^*$. By the previous proposition, the polar decomposition of the invertible element $\phi(a) = \phi(u)\phi(d)$ is unique. □

The functional calculus and polar decomposition are key results used in the next section to establish projectivity or semiprojectivity of certain C*-algebras.

4 Semiprojectivity

The established equivalences relating topological spaces and C*-algebras can now be implemented to establish operator algebra analogs to topological results for retracts.

4.1 Absolute Retracts and Absolute Neighborhood Retracts

The C*-algebra concept of semiprojectivity is closely related to the topological concept of absolute neighborhood retract, and equivalences from preceding consequences allow the relationship to be made precise. Projectivity and semiprojectivity are defined in a later subsection, where the specific relationships of ARs to projectivity and ANRs to semiprojectivity are established.

Let Y be a topological space with subspace Z. Then Z is a *retract* of Y if a continuous function $r : Y \to Z$ exists such that $r(z) = z$ for all $z \in Z$. A normal space X is an *absolute retract (AR)* if for every normal space Y and closed subspace Z of Y homeomorphic to X, Z is a retract of Y. A topological space X possesses the *universal extension property* if for every normal space Y, closed subspace Z of Y, and continuous function $f : Z \to X$, f extends to a continuous function $\tilde{f} : Y \to X$. The following proposition establishes the equivalence of the universal extension property and the AR property for compact Hausdorff spaces.

Example. The space $\{0, 1\}$ is not an absolute retract. Consider $\{0, 1\}$ as a subspace of $[0, 1]$ and let $f : \{0, 1\} \to \{0, 1\}$ be defined by $f(0) = 0$ and $f(1) = 1$. f cannot be extended to a continuous function over $[0, 1]$.

Proposition 4.1. *Let X be a compact Hausdorff space. Then X has the universal extension property if and only if X is an absolute retract.*

Proof. Suppose first that X has the universal extension property. Let Y be a normal space and let $Z \subseteq Y$ be any closed subspace homeomorphic to X with $f : Z \to X$ a homeomorphism between X and Z. Then f extends to a continuous function $\tilde{f} : Y \to X$. Define $r = f^{-1} \circ \tilde{f}$. The function r is continuous, and for any $z \in Z$, the image of z under both f and $\tilde{f}$ is $f(z)$, so

$$r(z) = f^{-1} \circ \tilde{f}(z) = f^{-1}(f(z)) = z$$

for all $z \in Z$. Hence Z is a retract of Y and X is an absolute retract.

Next suppose that X is an absolute retract. Let Y be a normal space, Z a closed subspace of Y, and $f : Z \to X$ a continuous function. Since X is compact Hausdorff, X is homeomorphic to some subspace X_0 of $[0, 1]^{\mathbb{N}}$. The space $[0, 1]$ possesses the universal extension property by the Tietze extension theorem. Consequently, $[0, 1]^{\mathbb{N}}$ also possesses the universal extension property. Let $g : X \to X_0$ be the homeomorphism between X and X_0 and define $h = g \circ f : Z \to X_0$. This continuous function h extends to a continuous function $\tilde{h} : Y \to [0,1]^{\mathbb{N}}$ since $[0,1]^{\mathbb{N}}$ has the universal extension property. Also, since X is an AR, X_0 is a retract of $[0,1]^{\mathbb{N}}$ so a continuous function $r : [0,1]^{\mathbb{N}} \to X_0$ exists such that $r(x) = x$ for all $x \in X_0$. Finally, define $\tilde{f} = g^{-1} \circ r \circ \tilde{h}$. Then $\tilde{f} : Y \to X$ is a continuous function extending f. □

Example. The compact Hausdorff space $[a,b]$, where $a, b \in \mathbb{R}$, has the universal extension property by the Tietze extension theorem. Hence by the preceding proposition, this space is an absolute retract.

A normal space X is an *absolute neighborhood retract (ANR)* if for every normal space Y, closed subspace Z of Y, and continuous function $f : Z \to X$, f extends to a continuous function $\tilde{f} : U \to X$ for some neighborhood $U \subseteq Y$ of Z. Any AR is automatically an ANR.

The concepts of AR and ANR can be restricted to a category of topological spaces. In the work following, unless otherwise noted X shall be an element of the category of compact metrizable spaces. In this case, X is an AR in the category of compact metrizable spaces if the conditions in the definition of AR above hold with the modification that any Y is restricted to be a compact metrizable space. Likewise, X is an ANR if the conditions for being an ANR hold for any compact metrizable space Y.

Lemma 4.2. *A space X is an ANR in the category of compact metrizable spaces if and only if for any compact metrizable space Y, decreasing sequence $\{Z_n\}_{n\in\mathbb{N}}$ of closed subsets of Y with $Z = \bigcap\limits_n Z_n$, and continuous function $f : Z \to X$, f extends to a continuous function $\tilde{f} : Z_n \to X$ for some sufficiently large $n \in \mathbb{N}$.*

Proof. Suppose that X is an ANR. Let Y be a compact metrizable space, $\{Z_n\}_n \subseteq Y$ a decreasing sequence of closed subsets of Y with $Z = \bigcap\limits_n Z_n$, and $f : Z \to X$ a continuous function. Since X is an ANR, a neighborhood $U \subseteq Y$ of Z exists such that f extends to a continuous function $\tilde{f} : U \to X$. $Y \backslash U$ is compact since U is open and Y is compact. Define $U_n = Y \backslash Z_n$ for each n. Then

$$\bigcup_n U_n = \bigcup_n Y \backslash Z_n = Y \cap \bigcup_n Z_n^c = Y \cap (\bigcap_n Z_n)^c = Y \backslash Z,$$

and since $Z \subseteq U$, $Y \backslash U \subseteq Y \backslash Z = \bigcup\limits_n U_n$. Hence $\{U_n\}_n$ is an open cover of the compact set $Y \backslash U$ so a finite subcover $\mathcal{U} = \{U_{n_1}, ..., U_{n_m}\} \subseteq \{U_n\}_n$ of $Y \backslash U$ exists. Since $\{Z_n\}$ is a decreasing sequence of closed sets, $\{U_n\}$ is an increasing sequence of open sets. In particular, $\mathcal{U}$ has a largest set U_{n_k}. Then $Y \backslash U \subseteq U_{n_k}$ so that $Z_{n_k} \subseteq U$. Since $Z_j \subseteq Z_{n_k}$ for all $j \geq n_k$ and since f extends to a continuous function $\tilde{f}$ on U, the restriction of $\tilde{f}$ to Z_j is a continuous extension of f to Z_j for all Z_j having $j \geq n_k$.

Conversely, suppose that for any compact metrizable space Y, decreasing sequence $\{Z_n\}_{n\in\mathbb{N}}$ of closed subsets of Y with $Z = \bigcap_n Z_n$, and continuous function $f : Z \to X$, there is an $N \in \mathbb{N}$ such that f extends to a continuous function $\tilde{f}_n : Z_n \to X$ for all $n \geq N$. Y is metrizable so let d be a metric on Y. Define a sequence of open neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ of Z by $U_n = \{y \in Y \mid d(Z, y) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$, and let $Z_n = \{y \in Y \mid d(Z, y) \leq \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then $\{Z_n\}_n$ is a decreasing sequence of closed subsets of Y such that $Z = \bigcap_n Z_n$. The definitions imply that $\overline{U_{n+1}} \subseteq Z_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Let $M \geq N$ and suppose that $f : Z \to X$ is a continuous function. By hypothesis, f extends to a continuous function $\tilde{f} : Z_M \to X$. Therefore since $U_M \subseteq \overline{U_M} \subseteq Z_M$ is a neighborhood of Z and $\tilde{f}\mid_{U_M} : U_M \to X$ is a continuous function extending f, X is an ANR in the category of compact metrizable spaces. $\square$

Example. S^1 is not an AR since there is no retract of D onto S^1; see [8] for a proof of this result. However, S^1 is an ANR.

4.2 Semiprojectivity and Partial Liftings

A separable C*-algebra A is *projective* if for any C*-algebra B, closed ideal J of B, and *-homomorphism $\phi : A \to B/J$, there is a *-homomorphism $\psi : A \to B$ such that $\phi = \pi \circ \psi$, where $\pi : B \to B/J$ is the natural quotient map. Any such ϕ is called *liftable*. A separable C*-algebra A is *semiprojective* if for any C*-algebra B, increasing sequence $\{J_n\}_n$ of closed ideals of B, and *-homomorphism $\phi : A \to B/J$, where $J = \overline{\bigcup_n J_n}$, there is an n and a *-homomorphism $\psi : A \to B/J_n$ such that $\phi = \pi_n \circ \psi$, where $\pi_n : B/J_n \to B/J$ is again the natural quotient map. Any such ϕ is called *partially liftable*. An element $a \in A$ is liftable if a homomorphism $\psi : A \to B$ exists such that $\phi(a) = \pi \circ \psi(a)$; a is partially liftable if a homomorphism $\psi : A \to B/J_n$ exists for some $n \in \mathbb{N}$ such that $\phi(a) = \pi_n \circ \psi(a)$.

The above definitions hold for the category of all C*-algebras. The definitions change slightly for the category of unital commutative C*-algebras: in this case, B is any unital commutative C*-algebra and both ϕ and ψ, provided the map ψ exists for the parameters specified, are unital *-homomorphisms. Similar adjustments modify definitions within the category of unital C*-algebras and the category of commutative C*-algebras.

The next theorem shows an equivalence between projective C*-algebras and absolute retracts.

Theorem 4.3. *Suppose that X is a compact metrizable space and $A = C(X)$ is a unital commutative C*-algebra. Then A is projective in the category of unital commutative C*-algebras if and only if X is an AR in the category of compact metrizable spaces.*

Proof. Suppose that X is an AR in the category of compact metrizable spaces. Let B be any unital commutative C*-algebra, $J \subseteq B$ a closed ideal, and $\phi : A \to B/J$ a unital *-homomorphism.

By the Gelfand representation $B \cong C(Y)$ for some compact metrizable space $Y = \Omega(A)$. Define $Z = \{y \in Y \mid g(y) = 0 \text{ for all } g \in J\}$; then $J \cong J_Y = \{g \in C(Y) \mid g(z) = 0 \text{ for all } z \in Z\}$. Moreover, $B/J \cong C(Y)/J_Y \cong C(Z)$ so some continuous function $f_\phi : Z \to X$ exists by the category equivalence between compact metrizable spaces and unital commutative C*-algebras. Since X is an AR, the function f_ϕ extends to a continuous function $\tilde{f}_\phi : Y \to X$. Again by the category equivalence, there is a *-homomorphism $\phi_{\tilde{f}} : A \to B$ such that $\phi = \pi \circ \phi_{\tilde{f}}$. Thus A is projective in the category of unital commutative C*-algebras.

Conversely, let $A = C(X)$ be a C*-algebra projective in the category of unital commutative C*-algebras. Also, suppose that Y is a compact metrizable space, $Z \subseteq Y$ is a closed subset, and $f : Z \to X$ is a continuous function. Then $J = \{g \in C(Y) \mid g(z) = 0 \text{ for all } z \in Z\}$ is a closed ideal in $B = C(Y)$. $B/J \cong C(Z)$ so a *-homomorphism $\phi_f : A = C(X) \to C(Z) \cong B/J$ exists corresponding to f. Consequently, a *-homomorphism $\psi : A = C(X) \to C(Y)$ exists such that $\phi_f = \pi \circ \psi$ since A is projective. There is a continuous function $f_\psi : Y \to X$ extending f corresponding to ψ by the category equivalence. Therefore X is an AR in the category of compact metrizable spaces. □

A similar result holds for semiprojective C*-algebras and absolute neighborhood retracts, as shown in the next theorem.

Theorem 4.4. *Suppose that X is a compact metrizable space and $A = C(X)$ is a unital commutative C*-algebra. Then A is semiprojective in the category of unital commutative C*-algebras if and only if X is an ANR in the category of compact metrizable spaces.*

Proof. Suppose that X is an ANR in the category of compact metrizable spaces. Let B be a unital commutative C*-algebra, $\{J_n\}_n$ an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : A \to B/J$ a *-homomorphism. By the Gelfand representation $B \cong C(Y)$ for some compact metrizable space Y. Define $Z_n = \{y \in Y \mid g(y) = 0 \text{ for all } g \in J_n\}$ for each $n \in \mathbb{N}$; then $\{Z_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of closed subsets of Y and $J_n \cong J_{n_Y} = \{g \in C(Y) \mid g(z) = 0 \text{ for all } z \in Z_n\}$ for each $n \in \mathbb{N}$. The ideal J corresponds to $Z \subseteq Y$ for a closed subset Z defined similarly. A continuous function $f_\phi : Z \to X$ exists by the category equivalence between compact metrizable spaces and unital commutative C*-algebras because $B/J \cong C(Z)$. Then some $N \in \mathbb{N}$ exists such that the function $f_\phi$extends to a continuous function $\tilde{f}_\phi : Z_n \to X$ for all $n \geq N$ since X is an ANR. By the same category equivalence, a *-homomorphism $\phi_{\tilde{f}} : A \to B/J_n$ exists such that $\phi = \pi \circ \phi_{\tilde{f}}$ since $B/J_n \cong C(Z_n)$. Hence A is semiprojective in the category of unital commutative C*-algebras.

Conversely, suppose that A is semiprojective. Let Y be a compact metrizable space, $\{Z_n\}_n$ a decreasing sequence of closed subsets of Y such that $Z = \bigcap_n Z_n$, and $f : Z \to X$ a continuous function. Then $J = \{g \in C(Y) \mid g(z) = 0 \,\text{for all}\, z \in Z\}$ is a closed ideal in $B = C(Y)$ and $J_n = \{g \in C(Y) \mid g(z) = 0 \,\text{for all}\, z \in Z_n\}$ is a closed ideal in B for each $n \in \mathbb{N}$ such that $\{J_n\}_n$ is an increasing sequence of closed ideals. Moreover, $B/J \cong C(Z)$ so a *-homomorphism $\phi_f : A = C(X) \to C(Z) \cong B/J$ exists corresponding to f by the category equivalence. Since A is semiprojective and $B/J_n \cong C(Z_n)$, there is a *-homomorphism $\psi : A = C(X) \to C(Z_n) \cong B/J_n$ with the property $\phi_f = \pi_n \circ \psi$ for some $n \in \mathbb{N}$. Again by the category equivalence, a continuous function $f_\psi : Z_n \to X$ extending f exists. Then since $Z_m \subseteq Z_n$ for all $m \geq n$, f extends to a continuous function on Z_m for all $m \geq n$, where the continuous extension for a particular Z_m is $f_\psi\,|_{Z_m}$. Therefore X is an ANR in the category of compact metrizable spaces. □

Example. The compact metrizable space $[-1, 1]$ is an AR so $C([-1, 1])$ is projective in the category of unital commutative C*-algebras.

4.3 Important Liftings and Partial Liftings

In this section, A and B are a C*-algebras, $\phi : A \to B/J$ is a *-homomorphism,$\{J_n\}_n$ is an increasing sequence of closed ideals of B, and $J = \overline{\bigcup_n J_n}$. Also, let $\pi : B \to B/J$, $\pi_{0,n} : B \to B/J_n$, $\pi_{k,n} : B/J_k \to B/J_n$, $\pi_n : B/J_n \to B/J$, for $k, n \in \mathbb{N}$ with $k \leq n$, be the natural quotient maps.

Theorem 4.5. *Suppose that $q_A \in A$ is a projection. Then q_A is partially liftable to a projection.*

Proof. Let $q = \phi(q_A)$. The element q is a projection in B/J since q_A is a projection. Some element $y \in B$ exists such that $\pi(y) = q$ since π is surjective. Let $x = \frac{1}{2}(y + y^*)$. Then x is self-adjoint, $\pi(x) = \frac{1}{2}([\pi(y)]^* + \pi(y)) = q$, and $\pi(x - x^2) = q - q^2 = 0$ since π is a *-homomorphism and q is a projection. Moreover, $\left\|\pi(x - x^2)\right\| = 0$ because $\pi(x - x^2) = 0$. Some $n \in \mathbb{N}$ exists such that $\left\|\pi_{0,n}(x - x^2)\right\| < \frac{1}{4}$ since $\left\|\pi(b)\right\| = inf\left\|\pi_n(b)\right\|$ for any $b \in B$. Let $z = \pi_{0,n}(x)$, the image of x in B/J_n; z is self-adjoint because x is self-adjoint. $\pi(x - x^2) = z - z^2$ so by functional calculus, $\sigma(z - z^2) = \{\lambda - \lambda^2 \mid \lambda \in \sigma(z)\} \subseteq (-\frac{1}{4}, \frac{1}{4})$, which implies that $\sigma(z) \subseteq (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$. In particular, $\frac{1}{2} \notin \sigma(z)$. Let $f : (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2}) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & for\, t \in (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \\ 1 & for\, t \in (\frac{1}{2}, \frac{1+\sqrt{2}}{2}) \end{cases}$$

and set $p = f(z)$. $\sigma(p) = f(\sigma(z)) = \{0, 1\}$ since $\sigma(z) \subseteq (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$. f is a projection in $C(\sigma(z))$ since $f(t) = \overline{f(t)} = [f(t)]^2 = 0$ for $t \in (\frac{1-\sqrt{2}}{2}, \frac{1}{2})$ and $f(t) = \overline{f(t)} = [f(t)]^2 = 1$ for $t \in (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$. Consequently, $p = f(z)$ is a projection in B/J_n and

$$\pi(p) = \pi(f(z)) = f(\pi(z)) = f(q) = q,$$

since q is a projection. Let $\psi : A \to B/J_n$ be a *-homomorphism such that $\psi(q_A) = p$. Then $\phi(q_A) = \pi \circ \psi(q_A) = q$, so q_A is partially liftable to the projection p. □

Corollary 4.6. *Let $\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_m \in A$ be mutually orthogonal projections. Then $\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_m$ are partially liftable to mutually orthogonal projections.*

Proof. Let $q_k = \pi(\tilde{q}_k)$ for $1 \leq k \leq n$; q_k is a projection for $1 \leq k \leq m$. By the previous theorem, a projection $\tilde{p}_1 \in B/J_{n_1}$ exists such that $\pi_{n_1}(\tilde{p}_1) = q_1$ for some $n_1 \in \mathbb{N}$. Then

$$q_k \in (1 - q_1)B/J(1 - q_1)$$

for $2 \leq k \leq m$ and $(1 - \pi_{n_1,n}(\tilde{p}_1))B/J_n(1 - \pi_{n_1,n}(\tilde{p}_1))$ is a C*-subalgebra of B/J_n for each $n \geq n_1$. Again by the previous theorem, a projection

$$\tilde{p}_2 \in (1 - \pi_{n_1,n_2}(\tilde{p}_1))B/J_{n_2}(1 - \pi_{n_1,n_2}(\tilde{p}_1))$$

exists such that $\pi_{n_2}(\tilde{p}_2) = q_2$ for some $n_2 \geq n_1$. Moreover, $\tilde{p}_2 \perp \pi_{n_1,n_2}(\tilde{p}_1)$. A third application of this theorem yields a projection

$$\tilde{p}_3 \in (1 - \pi_{n_1,n_3}(\tilde{p}_1) - \pi_{n_2,n_3}(\tilde{p}_2))B/J_{n_3}(1 - \pi_{n_1,n_3}(\tilde{p}_1) - \pi_{n_2,n_3}(\tilde{p}_2))$$

for some $n_3 \geq n_2$ such that $\tilde{p}_3$, $\pi_{n_1,n_3}(\tilde{p}_1)$, and $\pi_{n_2,n_3}(\tilde{p}_2)$ are mutually orthogonal. Continuing in this way, some $n_m \in \mathbb{N}$ exists such that $\tilde{p}_m$ is a projection in B/J_{n_m}while $\tilde{p}_m$ and the images of each $\tilde{p}_k$ in B/J_{n_m} for $1 \leq k \leq m-1$ are all mutually orthogonal and orthogonal to $\tilde{p}_k$. Let $p_k = \pi_{n_k,n_m}(\tilde{p}_k)$ and define a map $\psi : A \to B/J_{n_m}$ by $\psi(\tilde{q}_k) = p_k$ for $1 \leq k \leq n$. Then ψ is a *-homomorphism and $\phi(\tilde{q}_k) = \pi_{n_m} \circ \psi(\tilde{q}_k)$ for $1 \leq k \leq n$ and the projections $p_1, p_2, ..., p_m$ are mutually orthogonal. □

A unitary in a unital C*-algebra is not liftable but the following theorem shows that it is partially liftable.

Theorem 4.7. *Assume A and B are unital, and let $v_A \in A$ be a unitary. Then v_A is partially liftable to a unitary.*

Proof. Let $v = \phi(v_A)$. Since 1 is a projection in A, $\phi(1) = q$ for some projection $q \in B/J$. Then

$$v^*v = \phi(v_A^*)\phi(v_A) = \phi(v_A^* v_A) = \phi(1) = q \text{ and}$$

$$vv^* = \phi(v_A)\phi(v_A^*) = \phi(v_A v_A^*) = \phi(1) = q,$$

which shows that v is a partial isometry with source projection and range projection both equal to q. By the previous proposition, a projection $p \in B/J_n$ exists such that $\pi(p) = q$ for some $n \in \mathbb{N}$. Then $p(B/J_n)p = \{pxp \mid x \in B/J_n\}$ is a unital C*-subalgebra of B/J_n, $\pi_{n,m}(p(B/J_n)p)$ $(= \pi_{n,m}(p)(B/J_m)\pi_{n,m}(p))$ is a unital C*-subalgebra of B/J_m for all $m \geq n$, and $q(B/J)q$ is a unital C*-subalgebra of B/J. Moreover, $\phi : A \to q(B/J)q$ is a unital *-homomorphism since q is the unit in $q(B/J)q$ and $\phi(1) = q$. Thus v is a unitary in $q(B/J)q$.

Take $x \in p(B/J)p$ with $\pi_n(x) = v$. Then $\pi_{n,m}(x) \in \pi_{n,m}(p(B/J_n)p)$ for all $m \geq n$ and

$$\pi_n(x^*x) = v^*v = q. \ \left\|\pi_n(p - x^*x)\right\| = \left\|q - v^*v\right\| = 0,$$

so $m \in \mathbb{N}$ exists such that $\left\|\pi_{n,m}(p) - \pi_{n,m}(x^*x)\right\| < 1$ since the norm on $q(B/J)q$ is the infimum of the norms on $\pi_{n,m}(p(B/J_n)p)$. Let $y = \pi_{n,m}(x)$. Since

$$\left\|\pi_{n,m}(p) - \pi_{n,m}(x^*x)\right\| = \|\pi_{n,m}(p) - y^*y\| < 1$$

and $\pi_{n,m}(p)$ is the unit in $\pi_{n,m}(p(B/J_n)p)$, $y^*y = \pi_{n,m}(p) - (\pi_{n,m}(p) - y^*y)$ is invertible. By similar reasoning, yy^* is invertible. Consequently, y is itself invertible. By polar decomposition, a unitary u and a positive element d exist such that $y = ud$, where u is unitary in the sense that $u^*u = uu^* = \pi_{n,m}(p)$. Then

$$\pi(y) = \pi(u)\pi(d) = \pi(u)1 = v,$$

where $\pi(u) = v$ and $\pi(d) = 1$ by uniqueness of the polar decomposition of v in B/J. Let $\psi : A \to \pi_{n,m}(p(B/J_n)p)$ be a unital *-homomorphism such that $\psi(v_A) = u$. Then ψ is a homomorphism from A into B/J_m and $\phi(v_A) = \pi \circ \psi(v_A) = v$. Hence$v_A$ is partially liftable. □

The following proposition, needed in the theorem following, establishes that two projections having a normed difference less than 1 are unitarily equivalent. It also shows, in particular, that the functional calculus is preserved under *-homomorphism on a unitary relating two such projections. See [3] for a proof of the result.

Proposition 4.8. *Suppose that A is a unital C*-algebra and $p_1, p_2 \in A$ are projections such that $\|p_1 - p_2\| < 1$. Then there is a unitary $v = v(p_1, p_2) \in A$ such that $p_2 = vp_1v^*$. Moreover, $v(p, p) = 1$ for any projection p and the map $\theta : A \times A \to A$ defined by $\theta((p_1, p_2)) = v(p_1, p_2)$ is functorial in the sense that if $\phi : A \to B$ is a *-homomorphism between the unital C*-algebras A and B, then $\phi(v(p_1, p_2)) = v(\phi(p_1), \phi(p_2))$.*

The following theorem establishes that a partial isometry can be partially lifted in a way that preserves its source and range projections.

Theorem 4.9. *Suppose that $s_A \in A$ is a partial isometry, and let $s = \phi(s_A)$ have specified partial liftings p_1 of the source projection $q_1 = s^*s$ and p_2 of the range projection $q_2 = ss^*$ in B/J_m for some $m \in \mathbb{N}$. Then s_A is partially liftable to a partial isometry r in B/J_n for some $n \in \mathbb{N}$ with the properties that $\pi_n(r) = s$, $r^*r = \pi_{m,n}(p_1)$, and $rr^* = \pi_{m,n}(p_2)$.*

Proof. The element s is a partial isometry in B/J with source and range projections $q_1 = \phi(s_A^*s_A) = s^*s$ and $q_2 = \phi(s_As_A^*) = ss^*$. A prior theorem of this section establishes that projections are partially liftable, so both q_1 and q_2 can be lifted to projections p_1, p_2 in B/J_m such that $\pi_m(p_1) = q_1$ and $\pi_m(p_2) = q_2$ for some $m \in \mathbb{N}$.

Next, let $x \in B$ be an element such that $\pi(x) = s$. The reasoning of the proof above establishing that any projection is partially liftable implies that $\left\|\pi_{0,n}(x^*x - (x^*x)^2)\right\| < \frac{1}{4}$ in B/J_n for some $n \in \mathbb{N}$ since $\left\|\pi(x^*x - (x^*x)^2)\right\| = 0$. Moreover, letting $y = \pi_{0,n}(x)$, subsequent reasoning in the aforementioned proof leads to $\sigma(y^*y) \subseteq (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$.

Define a function $f : (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2}) \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & for\ t \in (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \\ t^{-\frac{1}{2}} & for\ t \in (\frac{1}{2}, \frac{1+\sqrt{2}}{2}) \end{cases}.$$

Let $z = yf(y^*y)$. Then $z^*z = f(y^*y)g(y^*y)f(y^*y)$, where $g : (\frac{1-\sqrt{2}}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1+\sqrt{2}}{2}) \to \mathbb{R}$ is defined by $g(t) = t$. Since $f(t)g(t)f(t) = 0$ for $t \in (\frac{1-\sqrt{2}}{2}, \frac{1}{2})$ and $f(t)g(t)f(t) = t^{-\frac{1}{2}}tt^{-\frac{1}{2}} = 1$ for $t \in (\frac{1}{2}, \frac{1+\sqrt{2}}{2})$, the function $h = fgf \in C(\sigma(y^*y))$ is a projection. Consequently, z^*z is a projection by the functional calculus at y^*y so the element z is a partial isometry in B/J_n. In particular, $zz^*$is also a projection. Let $\tilde{p}_1 = z^*z$ and $\tilde{p}_2 = zz^*$. Then since $\pi_n(y) = \pi(x) = s$, by properties of the functional calculus,

$$\pi_n(z) = \pi_n(yf(y^*y)) = \pi_n(y)f(\pi_n(y^*y)) = sf(s^*s) = ss^*s = s$$

since s is a partial isometry. Furthermore,

$$\begin{aligned} \pi_n(\tilde{p}_1) &= \pi_n(z^*z) = \pi_n(f(y^*y)g(y^*y)f(y^*y)) = f(\pi_n(y^*y))g(\pi_n(y^*y))f(\pi_n(y^*y)) \\ &= f(s^*s)g(s^*s)f(s^*s) = s^*ss^*ss^*s = s^*s = q_1, \text{ and} \\ \pi_n(\tilde{p}_2) &= \pi_n(zz^*) = \pi_n(yf(y^*y)f(y^*y)y^*) = \pi_n(y)f(\pi_n(y^*y))f(\pi_n(y^*y))\pi_n(y^*) \\ &= sf(s^*s)f(s^*s)s^* = ss^*ss^*ss^* = ss^* = q_2 \end{aligned}$$

Hence $\|\pi_n(\tilde{p}_1) - \pi_m(p_1)\| = 0$ in B/J, meaning $\|\pi_{n,N_1}(\tilde{p}_1) - \pi_{m,N_1}(p_1)\| < 1$ in B/J_{N_1} for some $N_1 \in \mathbb{N}$. Similarly, $\|\pi_{n,N_2}(\tilde{p}_2) - \pi_{m,N_2}(p_2)\| < 1$ in B/J_{N_2} for some $N_2 \in \mathbb{N}$. For the sake of simplicity in notation, henceforth $\tilde{p}_1$ shall denote $\pi_{n,N}(\tilde{p}_1)$ and p_1 shall denote $\pi_{n,N}(p_1)$; this convention shall apply to the denotations of $\tilde{p}_2$ and p_2 as well. Finally, z shall denote $\pi_{n,N}(z)$. Thus $\|\tilde{p}_1 - p_1\| < 1$ and $\|\tilde{p}_2 - p_2\| < 1$ in B/J_N. By the proposition preceding this theorem, there are unitaries $v_1 = v(p_1, \tilde{p}_1)$ and $v_2 = v(p_2, \tilde{p}_2)$ such that $\tilde{p}_1 = v_1 p_1 v_1^*$ and $\tilde{p}_2 = v_2 p_2 v_2^*$ in B/J_N.

Next, let $r = v_2^* z v_1$. Then

$$\begin{aligned} r^*r &= v_1^* z^* v_2 v_2^* z v_1 = v_1^* z^* z v_1 = v_1^* \tilde{p}_1 v_1 = p_1, \text{ and} \\ rr^* &= v_2^* z v_1 v_1^* z^* v_2 = v_2^* z z^* v_2 = v_2^* \tilde{p}_2 v_2 = p_2. \end{aligned}$$

Finally, by the functorial property of the functions $v_1 = v(p_1, \tilde{p}_1)$ and $v_2 = v(p_2, \tilde{p}_2)$,

$$\begin{aligned} \pi_N(r) &= \pi_N(v_2^* z v_1) = \pi_N([v(p_2, \tilde{p}_2)]^*)\pi_N(z)\pi_N(v(p_1, \tilde{p}_1)), \\ &= [v(\pi_N(p_2), \pi_N(\tilde{p}_2))]^*\pi_N(z)v(\pi_N(p_1), \pi_N(\tilde{p}_1)) = [v(q_2, q_2)]^* s v(q_1, q_1) = s \end{aligned}$$

since $v(p, p) = 1$ for any projection $p \in B/J$.

Therefore r is a partial isometry in B/J_N with source projection p_1 and range projection p_2, so r is a partial lifting of s in B/J_N with source and range projections that map to the source and range projections of s in B/J. $\square$

The following result concerning an isometry in a C*-algebra is obtained as a special case of the previous theorem.

Corollary 4.10. *Assume A is unital, and let $y_A \in A$ be an isometry such that $y = \phi(y_A)$ has specified partial liftings p_1 of the source projection $q_1 = y^*y$ and p_2 of the range projection $q_2 = yy^*$ in B/J_m for some $m \in \mathbb{N}$. Then y_A is partially liftable to a partial isometry z in B/J_n for some $n \in \mathbb{N}$ with the properties that $\pi_n(z) = y$, $z^*z = \pi_{m,n}(p_1)$, and $zz^* = \pi_{m,n}(p_2)$.*

Proof. Let $y = \phi(y_A)$, and let $q_1 = \phi(y_A^* y_A) = y^*y$ and $q_2 = \phi(y_A y_A^*) = yy^*$. By the previous theorem, y_A is partially liftable to a partial isometry $z \in B/J_n$ for some $n \in \mathbb{N}$ such that $\pi_n(z^*z) = y^*y$ and $\pi_n(zz^*) = yy^*$. A *-homomorphism $\psi : A \to B/J_n$ exists such that $\psi(y_A) = z$, meaning $\phi(y_A) = \pi_n \circ \psi(y_A)$. Furthermore, $\pi_n \circ \psi(y_A^* y_A) = y^*y$ and $\pi_n \circ \psi(y_A y_A^*) = yy^*$. Therefore y_A is partially liftable to a partial isometry preserving the source and range projections of y_A. $\square$

4.4 Semiprojective C*-Algebras

Many C*-algebras can be defined by a set of generators and a set of relations on those generators. Relations establish relationships between the generators and usually take the form of algebraic relations. Universal C*-algebras are an example of C*-algebras that can be defined in terms of a set of relations on a set of generators. Let $\mathcal{G} = \{x_i \mid i \in \Omega\}$ and let $\mathcal{R}$ be a set of relations. Suppose A is a C*-algebra such that

1. A is generated by a set of elements $\mathcal{Y} = \{y_i \mid i \in \Omega\}$ satisfying the relations $\mathcal{R}$.
2. If C is any C*-algebra with elements $\mathcal{Z} = \{z_i \mid i \in \Omega\}$ satisfying the relations $\mathcal{R}$, there is a *-homomorphism $\varphi : A \to C$ such that $\varphi(y_k) = z_k$ for all $k \in \Omega$.

Then $C^*(\mathcal{G} \mid \mathcal{R}) \cong A$ is the *universal C*-algebra* generated by $\mathcal{G}$ with relations $\mathcal{R}$.

Using results established in the previous section, showing that certain basic C*-algebras are semiprojective can be accomplished by recognizing that these C*-algebras can be defined as particular universal C*-algebras. The next propositions illustrate this fact. Below, $\mathbb{C}$ is shown to be semiprojective as the universal C*-algebra generated by a single projection.

Proposition 4.11. *Let*

$$\mathcal{G} = \{p\},\ \mathcal{R} = \{p = p^* = p^2\}.$$

Then $C^*(\mathcal{G} \mid \mathcal{R}) \cong \mathbb{C}$ is semiprojective.

Proof. $\mathbb{C}$ is generated by the element 1 satisfying the relations $\mathcal{R}$. Suppose that C is any other C*-algebra generated by a single projection p_C satisfying the relations $\mathcal{R}$. Then a *-homomorphism $\varphi : \mathbb{C} \to C$ mapping 1 to p_C exists since 1 is a projection that must map under any *-homomorphism to a projection in C and C contains the projection p_C. Therefore $\mathbb{C} \cong C^*(\mathcal{G} \mid \mathcal{R})$.

Let B be any C*-algebra, $\{J_n\}_n$ an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : \mathbb{C} \to B/J$ a *-homomorphism. Then 1 is partially liftable by Theorem 4.6, meaning a homomorphism $\psi : A \to B/J_n$ exists for some $n \in \mathbb{N}$ such that $\phi(1) = \pi_n \circ \psi(1)$. Therefore $\mathbb{C}$ is semiprojective since it is generated by the element 1. ☐

Next, the universal C*-algebra generated by a single unitary and a unit is shown to be $C(\mathbb{T})$, another semiprojective C*-algebra.

Proposition 4.12. *Let*

$$\mathcal{G} =\{u, 1\},\ \mathcal{R} =\{1 = 1^* = 1^2,\ u1 = 1u = u,\ u^*u = uu^* = 1\}.$$

Then $C^*(\mathcal{G} \mid \mathcal{R}) \cong C(\mathbb{T})$ is semiprojective.

Proof. The inclusion map ι is a unitary in the C*-algebra $C(\mathbb{T})$ since $|\iota(x)| = |x| = 1$ for all $x \in \mathbb{T}$. Moreover, the elements of $\mathcal{S} = \{\iota, 1\}$ satisfy the relations $\mathcal{R}$, and $\mathcal{S}$ generates $C(\mathbb{T})$ by the Stone-Weierstrass theorem. Let C be any other C*-algebra generated by element $\mathcal{Z} = \{z, 1_z\}$ satisfying the relations $\mathcal{R}$. Define a map $\varphi : C(\mathbb{T}) \to C$ by $\varphi(\iota) = z$ and $\varphi(1) = 1_z$. Then φ is a *-homomorphism so $C(\mathbb{T}) \cong C^*(\mathcal{G} \mid \mathcal{R})$.

Next, suppose that B is any C*-algebra, $\{J_n\}_n$ is an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : C(\mathbb{T}) \to B/J$ is a *-homomorphism. By theorem 4.8, 1 is partially liftable to a projection $p \in B/J_n$ and ι is partially liftable to an element $v \in B/J_n$ such that $v^*v = vv^* = p$ for some $n \in \mathbb{N}$. Moreover, p and v preserve the relations $\mathcal{R}$. Thus since $C(\mathbb{T})$ is a universal C*-algebra, a *-isomorphism $\psi : C(\mathbb{T}) \to B/J_n$ exists such that $\phi(\iota) = \pi_n \circ \psi(\iota)$ and $\phi(1) = \pi_n \circ \psi(1)$. Therefore $C(\mathbb{T})$ is semiprojective since ι and 1 generate $C(\mathbb{T})$. ☐

The Toeplitz algebra $\mathcal{T}$ is the universal C*-algebra generated by a single isometry. It is also semiprojective.

Proposition 4.13. *Let*

$$\mathcal{G} =\{y, 1\},\ \mathcal{R} =\{1 = 1^* = 1^2,\ y1 = 1y = y,\ y^*y = 1\}.$$

The Toeplitz algebra $\mathcal{T} = C^*(\mathcal{G} \mid \mathcal{R})$ is semiprojective.

Proof. Let B be any C*-algebra, $\{J_n\}_n$ an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : \mathcal{T} \to B/J$ a *-homomorphism. Let $q = \phi(1)$ and $s = \phi(y)$. Then q is a projection and

s is a partial isometry with source projection $s^*s = q$ and range projection ss^*. By theorem 5.5, the projections q and ss^* can be partially lifted to projections p and p_s, respectively, in B/J_{m_0} for some $m_0 \in \mathbb{N}$. Then by theorem 5.9, s can be lifted to a partial isometry $z \in B/J_m$ such that $z^*z = \pi_{m_o,m}(q)$ and $zz^* = \pi_{m_0,m}(p_s)$.

Then since $\mathcal{T}$ is a universal C*-algebra and $\mathcal{Z} = \{z, z^*z\}$ satisfies the relations R, a *-homomorphism $\psi : \mathcal{T} \to B/J_n$ exists such that $\phi(y) = \pi_m \circ \psi(y)$ and $\phi(1) = \pi_m \circ \psi(1)$. $\mathcal{T}$ is thus semiprojective since y and 1 generate $\mathcal{T}$. □

Another semiprojective C*-algebra is $M_n(\mathbb{C})$ for $n \in \mathbb{N}$ as shown below.

Proposition 4.14. *Let*

$$\mathcal{G} = \{e_{ij} \mid 1 \leq i, j \leq n\},\ \mathcal{R} = \{e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} | 1 \leq i, j, k, l \leq n\}$$

for $n \in \mathbb{N}$, *where* $\delta_{jk} = \begin{cases} 0 & if\ j \neq k \\ 1 & if\ j = k \end{cases}$. *Then* $C^*(\mathcal{G} \mid \mathcal{R}) \cong M_n(\mathbb{C})$ *is semiprojective.*

Proof. $M_n(\mathbb{C})$ is alternatively the universal C*-algebra generated by

$$\mathcal{G}_c = \{e_{1j} | 1 \leq j \leq n\},$$
$$\mathcal{R}_c = \{(e_{1j}^*e_{1j})^2 = e_{1j}^*e_{1j} = (e_{1j}^*e_{1j})^*, e_{1i}e_{1i}^* = e_{1j}e_{1j}^* = e_{11} \text{ for all } 1 \leq i, j \leq n,$$
$$e_{1i}^*e_{1i} \perp e_{1j}^*e_{1j} \text{ for all } i \neq j\}.$$

This is true because $e_{ij} \in \mathcal{G}$ can be retrieved as $e_{1i}^*e_{1j}$ for $e_{1i}, e_{1j} \in \mathcal{G}_c$ and each set of generators satisfy the same relations.

Let B be any C*-algebra, $\{J_n\}_n$ an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : M_n(\mathbb{C}) \to B/J$ a *-homomorphism, and let $f_{1j} = \phi(e_{1j})$ for $1 \leq j \leq n$. Then f_{1j} is a partial isometry with source projection $f_{1j}^*f_{1j}$ and range projection $f_{1j}f_{1j}^* = f_{11}$ for $1 \leq j \leq n$. Moreover, the n source projections are mutually orthogonal so by corollary 4.6, they can be lifted to n mutually orthogonal projections $p_1, p_2, ..p_n$ in B/J_{m_0} for some $m_0 \in \mathbb{N}$. By theorem 4.9, each partial isometry f_{1j} can be lifted to a partial isometry s_{1j} in B/J_m such that $s_{1j}^*s_{1j} = \pi_{m_0,m}(p_j)$ and $s_{1j}s_{1j}^* = \pi_{m_0,m}(p_1)$ for $1 \leq j \leq n$ and some $m \in \mathbb{N}$.

The partial isometries $\mathcal{S} = \{s_{1j} \mid 1 \leq j \leq n\}$ satisfy the relations $\mathcal{R}$ so since $M_n(\mathbb{C})$ is a universal C*-algebra, a *-homomorphism ψ exists such that $\phi = \pi_m \circ \psi$. Therefore $M_n(\mathbb{C})$ is semiprojective. □

The Cuntz algebra O_n, another semiprojective C*-algebra, is the universal C*-algebra generated by n isometries having mutually orthogonal range projections with sum equal to a unit 1. The Cuntz-Krieger algebra O_A on an matrix $n \times n$ matrix A generalizes the Cuntz algebra; it is semiprojective as well. When A consists of all entries $a_{ij} = 1$ for $1 \leq i, j \leq n$, then $O_A = O_n$.

Proposition 4.15. *Let*

$$\mathcal{G} = \{s_i, 1 \mid 1 \le i \le n\},$$
$$\mathcal{R} = \{1 = 1^* = 1^2,\ s_i 1 = 1 s_i = s_i,\ s_i^* s_i = 1,\ \sum_{j=1}^{n} s_j s_j^* = 1 | 1 \le i \le n\}$$

for $n \in \mathbb{N}$. The Cuntz algebra $O_n = C^*(\mathcal{G} \mid \mathcal{R})$ is semiprojective.

Proof. Let B be any C*-algebra, $\{J_n\}_n$ an increasing sequence of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and $\phi : O_n \to B/J$ a *-homomorphism. Let $r_i = \phi(s_i)$ for $1 \le i \le n$. Since 1 is a projection, there is a partial lifting of 1 to a projection $p \in B/J_{N_a}$ for some $N_a \in \mathbb{N}$ by theorem 4.5. Then by corollary 4.6, the mutally orthogonal projections $r_1 r_1^*, r_2 r_2^*, ..., r_n r_n^*$ are partially liftable to mutually orthogonal projections $p_1, p_2, ..., p_n$ in B/J_{N_b} for some $N_b \ge N_a$. Finally, by corollary 4.10 each isometry r_i is partially liftable to an isometry $z_i \in B/J_{m_i}$ having the properties $z_i^* z_i = \pi_{N_a,m}(p)$ and $z_i z_i^* = \pi_{N_b,m}(p_i)$ for $1 \le i \le n$ and some $m \in \mathbb{N}$.

Since O_n is a universal C*-algebra and set $\mathcal{Z} = \{z_i \mid 1 \le i \le n\}$ satisfies the relations $\mathcal{R}$, a *-homomorphism $\psi : O_n \to B/J_m$ exists such that $\psi(s_i) = z_i$ for $1 \le i \le n$. Moreover, $\phi = \pi_m \circ \psi$ so O_n is semiprojective. □

Corollary 4.16. *Let A be an $n \times n$ matrix for $n \in \mathbb{N}$ with $a_{ij} \in \{0,1\}$ for $1 \le i,j \le n$, and set*

$$\mathcal{G} = \{s_i, 1 | 1 \le i \le n\},$$
$$\mathcal{R} = \{1 = 1^* = 1^2,\ s_i 1 = 1 s_i = s_i,\ s_i^* s_i = \sum_{j=1}^{n} A_{ij} s_j s_j^*,\ s_k s_k^* \perp s_l s_l^* \mid 1 \le i,k,l \le n,\ k \ne l\}.$$

The Cuntz-Krieger algebra $O_A = C^*(\mathcal{G} \mid \mathcal{R})$ is semiprojective.

Proof. The proof of this corollary follows the same reasoning as the proof of the preceding proposition with the exception that the sum of the range projections depends on the matrix A. □

The Cuntz algebra O_∞ is the universal C*-algebra generated by a sequence of mutually orthogonal range projections. O_∞ is semiprojective although the theory establishing this fact is beyond the scope of this paper. See [3] for a construction demonstrating this result.

One last semiprojective C*-algebra is $C^*(\mathbb{F}^n)$, the universal C*-algebra generated by n unitaries.

Proposition 4.17. *Let*

$$\mathcal{G} = \{u_i, 1 \mid 1 \le i \le n\},$$
$$\mathcal{R} = \{1 = 1^* = 1^2,\ u_i 1 = 1 u_i = u_i,\ u_i^* u_i = u_i u_i^* = 1\}$$

for $n \in \mathbb{N}$. Then $C^*(\mathbb{F}^n) = C^*(\mathcal{G} \mid \mathcal{R})$ is semiprojective.

Proof. The universal C*-algebra $C^*(\mathcal{G} \mid \mathcal{R}) = C^*(\mathbb{F}^n)$ is unital with unit 1. Each $u_i \in C^*(\mathbb{F}^n)$ is a unitary, so by theorem 4.8, for any C*-algebra B, increasing sequence $\{J_n\}_n$ of closed ideals of B with $J = \overline{\bigcup_n J_n}$, and *-homomorphism $\phi : C^*(\mathbb{F}^n) \to B/J$, there is a *-homomorphism $\psi_i : C^*(\mathbb{F}^n) \to$

B/J_{m_i} such that $\psi_i(u_i)$ is a unitary and $\phi(u_i) = \pi_{m_i} \circ \psi_i(u_i)$ for some $m_i \in \mathbb{N}$. Let $m = \max_{1 \leq i \leq n} m_i$ and let $v_i = \pi_{m_i,m} \circ \psi_i(u_i)$; define a map $\psi : C^*(\mathbb{F}^n) \to B/J_m$ by $\psi(u_i) = v_i$. Then each v_i is a unitary in B/J_m, so ψ is a *-homomorphism such that $\phi = \pi_m \circ \psi$. Thus $C^*(\mathbb{F}^n)$ is semiprojective. □

Unlike O_∞, the C*-algebra $C^*(\mathbb{F}^\infty)$, the universal C*-algebra generated by a sequence of unitaries, is not semiprojective.

Moreover, if the unitaries in the construction $C^*(\mathcal{G} \mid \mathcal{R})$ of the previous proposition have additional relations requiring that the generators commute, the resulting universal C*-algebra may not be semiprojective. The C*-algebra $C(\mathbb{T}^n)$ is the universal C*-algebra generated by n commuting unitaries. And in fact, $C(\mathbb{T}^n)$ is not semiprojective whenever $n \geq 2$; see [3] for a discussion of this result.

5 Conclusion

The intent of this paper is to explore an equivalence between topologies and C*-algebras starting from the rudiments of C*-algebra theory as well as to derive meaningful results from the equivalence. To this end, preliminary theory delves into the substance of the character space of $C(X)$ and the space of maximal ideals on $C(X)$. From the ensuing homeomorphisms, a category equivalence is established between certain topologies and C*-algebras. This process results in a collection of semiprojective C*-algebras, illuminating certain useful aspects of these algebras.

The basic method for advancing theory in this paper is powerful but well-established. Galois theory, for example, makes use of a similar type of correspondence to yield conclusions concerning polynomials out of other areas of abstract algebra. In the case of operator algebra theory, often topology is the mathematical context from which important results can be carried over to deduce consequences for operator algebras. This paper explores one such set of correspondences, that of absolute retracts and absolute neighborhood retracts as they relate to projectivity and semiprojectivity. The process illustrates, among other results highlighted herein, that extensions of continuous functions in a topological context correspond to the existence of particular *-homomorphisms in a C*-algebra context. And while in this case, results from topology are borrowed to establish C*-algebra results, C*-algebra theory can also produce new results in topology. K-theory contains examples of transplanting theory in the opposite direction within the scope of general correspondences relating topology and operator algebras. Often in mathematics this process is fruitful.

Nonetheless, the analogy between C*-algebras and topological spaces is limited in scope. As intimated in the presentation of semiprojective C*-algebras, some C*-algebras are not directly connected to topological spaces by analogy similar to that employed in this paper. Alternative arguments without straightforward relation to topological arguments are sometimes needed to analyze the structure of these C*-algebras, many of which are natural candidates to evaluate for the property of semiprojectivity.

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